Modal Dependent Type Theory and Dependent Right Adjoints

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Outline

Many examples of modalities: nominal type theory, guarded and clocked type theory, and spatial and cohesive type theory.

Modal dependent type theory with an operator satisfying the K-axiom of modal logic.

$$\Box(A \to B) \to \Box A \to \Box B$$

We provide semantics and syntax for a DTT with universes.

Since every finite limit category with an adjunction of endofunctors gives rise to model, we call this *dependent right adjoint*

Dependent K

$$\Box(\Pi y : A.\, B) \to \Pi x : \Box A.\, \Box B[\mathsf{open}\, x/y]$$

Why HoTT?

Most of these modal type theories only have an extensional TT
Want: a normalizing calculus

Application of HoTT:
The best approach for a calculus with funext is CTT
For guarded type theory, we have provided GCTT
We hope this work will allow us to provide a simpler calculus than GCTT (which also includes $\Box$)

Application to HoTT:
CTT with streams (Vezossi)
Towards a general extension of HoTT with guarded (co-)recursion

Dependent right adjoint

A *dependent right adjoint* then extends the definition of CwF with a functor on contexts L and an operation on families R, intuitively understood to be left and right adjoints:

Definition (category with a dependent right adjoint)
A *CwDRA* is a CwF C equipped with the following extra structure:

1. An endofunctor $\mathsf{L} : \mathsf{C} \to \mathsf{C}$ on the underlying category.
2. For each object $\Gamma \in \mathsf{C}$ and family $A \in \mathsf{C}(\mathsf{L}\Gamma)$, a family $\mathsf{R}_\Gamma A \in \mathsf{C}(\Gamma)$, stable under re-indexing in the sense that for all $\gamma \in \mathsf{C}(\Delta, \Gamma)$ we have $(\mathsf{R}_\Gamma A)[\gamma] = \mathsf{R}_\Delta(A[\mathsf{L}\gamma]) \in \mathsf{C}(\Delta)$
3. For each object $\Gamma \in \mathsf{C}$ and family $A \in \mathsf{C}(\mathsf{L}\Gamma)$ a bijection

$$\mathsf{C}(\mathsf{L}\Gamma \vdash A) \cong \mathsf{C}(\Gamma \vdash \mathsf{R}_\Gamma A) \tag{1}$$

We write the effect of this bijection on $a \in \mathsf{C}(\mathsf{L}\Gamma \vdash A)$ as $\overline{a} \in \mathsf{C}(\Gamma \vdash \mathsf{R}_\Gamma A)$ and write the effect of its inverse on $b \in \mathsf{C}(\Gamma \vdash \mathsf{R}_\Gamma A)$ also as $\overline{b} \in \mathsf{C}(\mathsf{L}\Gamma \vdash A)$. Thus

$$\overline{\overline{a}} = a \qquad (a \in \mathsf{C}(\mathsf{L}\Gamma \vdash A)) \tag{2}$$

$$\overline{\overline{b}} = b \qquad (b \in \mathsf{C}(\Gamma \vdash \mathsf{R}_\Gamma A)) \tag{3}$$

The bijection is required to be stable under re-indexing.

Mode theory

Conjecture Licata:
Our works fits within the mode theory framework
Checked for STT

Dradjoint as a guide for the general case ?

Syntax

Context formation rules:

$$\frac{}{\diamond \vdash} \qquad \frac{\Gamma \vdash \quad \Gamma \vdash A}{\Gamma, x : A \vdash}\; x \notin \Gamma \qquad \frac{\Gamma \vdash}{\Gamma, \text{🔓} \vdash} \qquad \frac{\Gamma, x : A, y : B, \Gamma' \vdash}{\Gamma, y : B, x : A, \Gamma' \vdash}\; x \text{ NOT FREE IN } B$$

Type formation rules:

$$\frac{\Gamma \vdash A \quad \Gamma, x : A \vdash B}{\Gamma \vdash \Pi x : A.\, B} \qquad \frac{\Gamma, \text{🔓} \vdash A}{\Gamma \vdash \Box A}$$

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash A = B}{\Gamma \vdash t : B} \qquad \frac{\Gamma, x : A, \Gamma' \vdash}{\Gamma, x : A, \Gamma' \vdash x : A}\; \text{🔓} \notin \Gamma' \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : \Pi x : A.\, B}$$

$$\frac{\Gamma \vdash t : \Pi x : A.\, B \quad \Gamma \vdash u : A}{\Gamma \vdash t\, u : B[u/x]}$$

$$\frac{\Gamma, \text{🔓} \vdash t : A}{\Gamma \vdash \mathsf{shut}\ t : \Box A} \qquad \frac{\Gamma \vdash t : \Box A \quad \Gamma, \text{🔓}, \Gamma' \vdash}{\Gamma, \text{🔓}, \Gamma' \vdash \mathsf{open}\ t : A}\; \text{🔓} \notin \Gamma'$$

Term equality rules

$$\frac{\Gamma \vdash (\lambda x.t)u : A}{\Gamma \vdash (\lambda x.t)u = t[u/x] : A} \qquad \frac{\Gamma \vdash \mathsf{open}\ \mathsf{shut}\ t : A}{\Gamma \vdash \mathsf{open}\ \mathsf{shut}\ t = t : A} \qquad \frac{\Gamma \vdash t : \Pi x : A.\, B}{\Gamma \vdash t = \lambda x.t\, x : \Pi x : A.\, B}\; x \notin \Gamma$$

$$\frac{\Gamma \vdash t : \Box A}{\Gamma \vdash t = \mathsf{shut}\ \mathsf{open}\ t : \Box A}$$

Sytax and semantics

Sound interpretation
Term model (conjecture: initiality)
Normalization and canonicity for simply typed case (Clouston).
Conjecture: extends to dependent case.

Local universe

Definition

Let C be a cartesian category. The **Giraud CwF of** C ($\mathbb{G}$C) is the CwF whose underlying category is C, and where a family $A \in \mathbb{G}\mathrm{C}(\Gamma)$ is a pair of morphisms

$$\begin{array}{ccc} & & E \\ & & \downarrow{\scriptstyle v} \\ \Gamma & \xrightarrow[u]{} & U \end{array} \tag{4}$$

and an element is a map $a : \Gamma \to E$ such that $v \circ a = u$.

Reindexing

$$A[\gamma] \triangleq (u \circ \gamma, v) \in \mathbb{G}\mathrm{C}(\Delta)$$

$$a[\gamma] \triangleq a \circ \gamma \in \mathbb{G}\mathrm{C}(\Delta \vdash A[\gamma])$$

The comprehension $\Gamma.A \in \mathrm{C}$ is given by the pullback of diagram (4).

CwDRA from cartesian category

Definition

A **weak CwF morphism** R between CwFs consists of a functor $\mathsf{R} : \mathsf{C} \to \mathbf{C}D$ between the underlying categories preserving the terminal object, an operation on families mapping $A \in \mathsf{C}(\Gamma)$ to a family $\mathsf{R}A \in \mathbf{C}D(\mathsf{R}\Gamma)$, an operation on elements mapping $a \in \mathsf{C}(\Gamma \vdash A)$ to an element $\mathsf{R}a \in \mathbf{C}D(\mathsf{R}\Gamma \vdash \mathsf{R}A)$, and an isomorphism $\nu_{\Gamma,A} : \mathsf{R}\Gamma.\mathsf{R}A \to \mathsf{R}(\Gamma.A)$, inverse to $(\mathsf{Rp}_A, \mathsf{Rq}_A)$. Required to commute with reindexing: $\mathsf{R}A[\mathsf{R}\gamma] = \mathsf{R}(A[\gamma])$ and $\mathsf{R}t[\mathsf{R}\gamma] = \mathsf{R}(t[\gamma])$.

CwFs as discrete comprehension categories and then using pseudo-maps of comprehension cats.

Theorem

$\mathbb{G}$ is a (fully faithful) functor from the category of cartesian categories and finite limit preserving functors, to the category of CwFs with weak morphisms.

Theorem

If C is a cartesian category and $\mathsf{L} \dashv \mathsf{R}$ are adjoint endofunctors on C, then $\mathbb{G}\mathsf{C}$ has the structure of a CwDRA.

Examples

- Nominal sets
- Guarded and Clocked Type Theory
- Cohesive toposes
- $\mathbb{I}$ is *tiny* if exponentation by it has a right-adjoint $\surd$.
  Dependent right-adjoint plays is important in the construction of the universe in cubical sets.
  (Licata, Orton, Pitts, Spitters)

Universes

Extension to universes using Coquand's CwU:
Presheaf models interpret an *inverse* $\ulcorner\urcorner$ to El from codes to types.

Let $\tilde{\mathsf{U}} \to \mathsf{U}$ be a universe.
Suppose that R preserves small fibers.
Then R can be lifted to a CwDRA with universes.
The image under R of maps with U-small fibers is classified by the universe with codes RU.

Examples include essential geometric morphisms given by functors on the underlying category.

Convenient universe polymorphic category seems to be missing.

HoTT

When the modal operator preserves fibrations,
this extends to HoTT

Example: GCTT

Conclusions

Dependent right-adjoint

- Syntax, semantics
- Many examples from the literature and from adjunction on lex category.