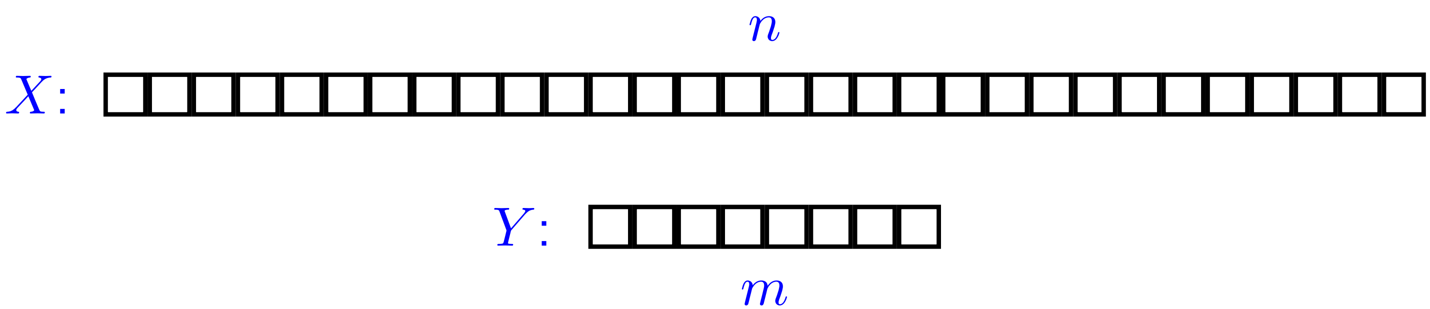
Comparison Based Merging

Upper and Lower bounds

Merging

Input: Two sorted lists X and Y of length n and m.

We may assume $n \geq m$.



Theorem:

In a comparison based model, the complexity of merging X and Y is

$$\Theta(m(\log(n/m) + 1))$$

Simple Upper Bounds

Standard Merge:

$$\Theta(n+m)$$

Binary Insertion of Y in X:

$$\Theta(m \log n))$$

For "large" m ($m = \Theta(n)$):

$$\Theta(n+m) = \Theta(m(\log(n/m)+1))$$

For "small" m (e.g. $m = O(\sqrt{n})$):

$$\Theta(m \log n) = \Theta(m(\log(n/m)+1))$$

The Simple Bounds are Sub-Optimal

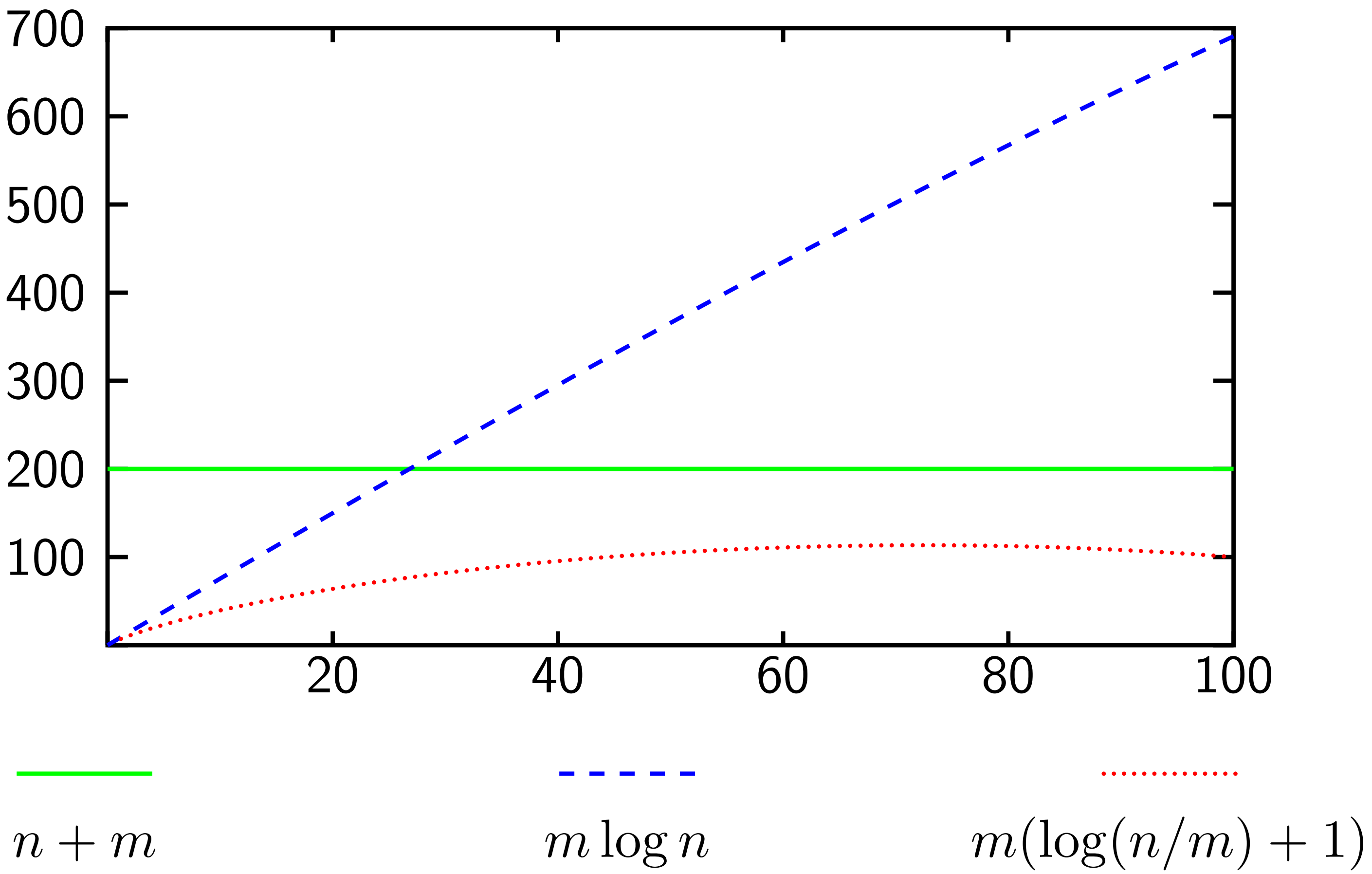
E.g. for $m = \Theta(n/\log n)$:

$$\Theta(n + m) = \Theta(n)$$

$$\Theta(m \log n) = \Theta(n)$$

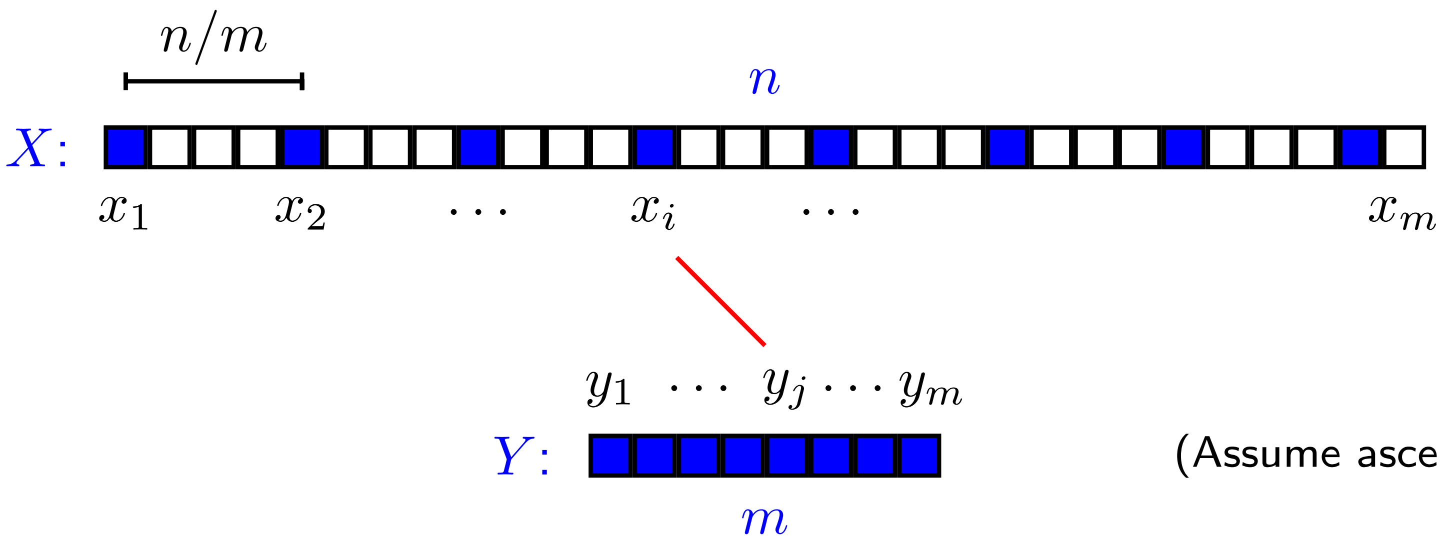
$$\Theta(m(\log(n/m) + 1)) = \Theta(n \frac{\log \log n}{\log n}) = o(n)$$

Graphically



$$\boxed{n + m = 200}$$

Better Upper Bound



(Assume ascending order)

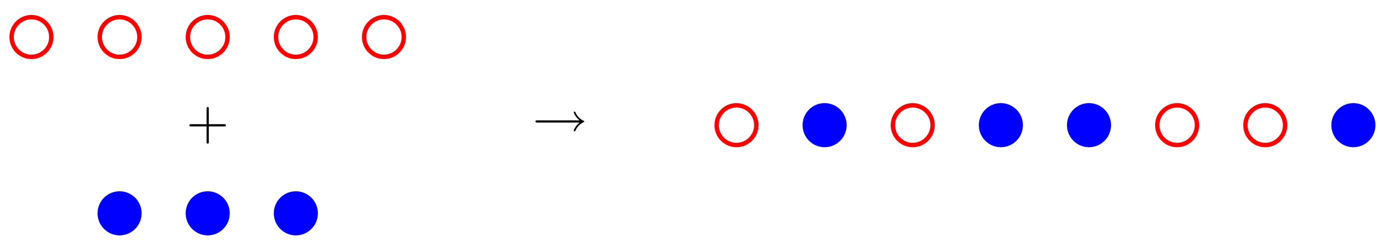
if $x_i < y_j$
  $i{+}{+}$
else
  binary search from x_{i-1} to x_i
  $j{+}{+}$

Number of comparisons:

$$m + m\log(n/m)$$

Lower Bound

There are $\binom{n+m}{m}$ different possible results of the merging two sorted lists of lengths n and m.



So any decision tree for merging must have at least that many leaves.

It must hence have height at least

$$\log(\binom{n+m}{m})$$

Lemmas

For $n \geq m$:

1)

$$\binom{n+m}{m} = \frac{(n+m)(n+m-1)\cdots(n+1)}{m(m-1)\cdots 1} \geq (n/m)^m$$

2)

$$\binom{n+m}{m} \geq \binom{2m}{m} \geq \frac{2m(2m-1)\cdots(m+1)}{m(m-1)\cdots 1}$$

$$\geq 2(\frac{m}{m})2(\frac{m-1/2}{m-1})2(\frac{m-2/2}{m-2})2(\frac{m-3/2}{m-3})\cdots \geq 2^m$$

Lemmas

3)

$$h(n) \geq f(n) \text{ and } h(n) \geq g(n)$$

$$\Updownarrow$$

$$h(n) \geq \max\{f(n), g(n)\}$$

4)

For f and g positive:

$$\max\{f(n), g(n)\} = \Theta(f(n) + g(n))$$

Lower Bound Computation

$$\log(\binom{n+m}{m})$$

$$\geq \max\{\log(2^m), \log((n/m)^m)\}$$

$$= \max\{m, m\log(n/m)\}$$

$$= \Omega(m + m\log(n/m)\})$$