On Grids in Topological Graphs

A *topological graph* is a graph drawn in the plane with vertices represented by points and edges as arcs connecting its vertices. If the edges are drawn as straight-line segments, then the graph is *geometric*. A \((k, l)\)-grid in a topological graph is a pair of edge subsets \(E_1\) and \(E_2\), such that \(|E_1| = k\), \(|E_1| = l\), and every edge in \(E_1\) crosses every edge in \(E_2\). It is known that for fixed constants \(k, l\), every \(n\)-vertex topological graph with no \((k, l)\)-grid has \(O(n)\) edges. We conjecture that this remains true even when: (1) considering grids with distinct vertices; or (2) the edges within each subset of the grid are required to be *pairwise disjoint* and the graph is geometric. These conjectures are shown to be true apart from \(\log^* n\) and \(\log^2 n\) factors, respectively. We also settle the second conjecture for the first nontrivial case \(k = 2, l = 1\), and for convex geometric graphs. The latter result follows from a stronger statement that generalizes the celebrated Marcus-Tardos Theorem on excluded patterns in 0-1 matrices.