Dynamic programming

- memoization
- decorator memoized
- systematic subproblem computation

Binomial coefficient

$$\binom{n}{k} = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = n \\ \binom{n-1}{k} + \binom{n-1}{k-1} & \text{otherwise} \end{cases}$$

```
binomial_recursive.py
```

```python
def binomial(n, k):
    if k == 0 or k == n:
        return 1
    return binomial(n - 1, k) + binomial(n - 1, k - 1)
```

```
--(7, 5)
  |--(6, 5)
  |   |--(5, 5)
  |    --(5, 4)
  |       |--(4, 4)
  |        --(4, 3)
  |           |--(3, 3)
  |            --(3, 2)
  |               |--(2, 2)
  |                --(2, 1)
  |                   |--(1, 1)
  |                    --(1, 0)
   --(6, 4)
      |--(5, 4)
      |   |--(4, 4)
      |    --(4, 3)
      |       |--(3, 3)
      |        --(3, 2)
      |           |--(2, 2)
      |            --(2, 1)
      |               |--(1, 1)
      |                --(1, 0)
       --(5, 3)
          |--(4, 3)
          |   |--(3, 3)
          |    --(3, 2)
          |       |--(2, 2)
          |        --(2, 1)
          |           |--(1, 1)
          |            --(1, 0)
           --(4, 2)
              |--(3, 2)
              |   |--(2, 2)
              |    --(2, 1)
              |       |--(1, 1)
              |        --(1, 0)
               --(3, 1)
                  |--(2, 1)
                  |   |--(1, 1)
                  |    --(1, 0)
                   --(2, 0)
```

identical computations

recursion tree for `binomial(7,5)`

Dynamic Programming
≡
Remember solutions already found (memoization)

- Technique sometimes applicable when running time otherwise becomes exponential
- Only applicable if stuff to be remembered is manageable

Binomial Coefficient
Dynamic programming using a dictionary

recursion tree for `binomial(7,5)`

```
--(7, 5)
   |--(6, 5)
   |   |--(5, 5)
   |    --(5, 4)
   |       |--(4, 4)
   |        --(4, 3)
   |           |--(3, 3)
   |            --(3, 2)
   |               |--(2, 2)
   |                --(2, 1)
   |                   |--(1, 1)
   |                    --(1, 0)
    --(6, 4)
       |--(5, 4)
        --(5, 3)
           |--(4, 3)
            --(4, 2)
               |--(3, 2)
                --(3, 1)
                   |--(2, 1)
                    --(2, 0)
```

reuse value stored in dictionary `answers`

binomial_dictionary.py

```python
answers = {}  # answers[(n, k)] = binomial(n,k)

def binomial(n, k):
    if (n, k) not in answers:
        if k==0 or k==n:
            answer = 1
        else:
            answer = binomial(n-1, k) + binomial(n-1, k-1)
        answers[(n, k)] = answer
    return answers[(n,k)]
```

Python shell

```
> binomial(6, 3)
| 20
> answers
| {(3, 3): 1, (2, 2): 1, (1, 1): 1, (1, 0): 1, (2, 1): 2, (3, 2):
  3, (4, 3): 4, (2, 0): 1, (3, 1): 3, (4, 2): 6, (5, 3): 10, (3,
  0): 1, (4, 1): 4, (5, 2): 10, (6, 3): 20}
```

- Use a dictionary `answers` to store already computed values

Question – What is the order of the size of the dictionary **answers** after calling **binomial(n,k)** ?

binomial_dictionary.py

```
answers = {}  # answers[(n, k)] = binomial(n,k)
def binomial(n, k):
    if (n, k) not in answers:
        if k==0 or k==n:
            answer = 1
        else:
            answer = binomial(n-1, k) + binomial(n-1, k-1)
        answers[(n, k)] = answer
    return answers[(n,k)]
```

a) max(n, k)
b) n + k
c) n * k
d) n^k
e) k^n
f) Don’t know

Binomial Coefficient
Dynamic programming using decorator

- Use a decorator (@memoize) that implements the functionality of remembering the results of previous function calls

binomial_decorator.py

```
def memoize(f):
    # answers[args] = f(*args)
    answers = {}

    def wrapper(*args):
        if args not in answers:
            answers[args] = f(*args)
        return answers[args]

    return wrapper
```

```
@memoize
def binomial(n, k):
    if k==0 or k==n:
        return 1
    else:
        return binomial(n-1, k) + binomial(n-1, k-1)
```

www.python-course.eu/python3_memoization.php

binomial_decorator_trace.py

```python
def trace(f):  # decorator to trace recursive calls
    indent = 0

    def wrapper(*args):
        nonlocal indent

        spaces = '|  ' * indent
        arg_str = ', '.join(map(repr, args))
        print(spaces + f'{f.__name__}({arg_str})')
        indent += 1
        result = f(*args)
        indent -= 1
        print(spaces + f'> {result}')
        return result

    return wrapper

def memoize(f):
    answers = {}

    def wrapper(*args):
        if args not in answers:
            answers[args] = f(*args)
        return answers[args]

    wrapper.__name__ = f.__name__ + '_memoize'

    return wrapper

@trace
@memoize
def binomial(n, k):
    if k == 0 or k == n:
        return 1
    return binomial(n - 1, k) + binomial(n-1, k-1)

print(binomial(5, 2))
```

Python shell (without @memoize)

```
binomial(5, 2)
|  binomial(4, 2)
|  |  binomial(3, 2)
|  |  |  binomial(2, 2)
|  |  |  > 1
|  |  |  binomial(2, 1)
|  |  |  |  binomial(1, 1)
|  |  |  |  > 1
|  |  |  |  binomial(1, 0)
|  |  |  |  > 1
|  |  |  > 2
|  |  > 3
|  |  binomial(3, 1)
|  |  |  binomial(2, 1)
|  |  |  |  binomial(1, 1)
|  |  |  |  > 1
|  |  |  |  binomial(1, 0)
|  |  |  |  > 1
|  |  |  > 2
|  |  |  binomial(2, 0)
|  |  |  > 1
|  |  > 3
|  > 6
|  binomial(4, 1)
|  |  binomial(3, 1)
|  |  |  binomial(2, 1)
|  |  |  |  binomial(1, 1)
|  |  |  |  > 1
|  |  |  |  binomial(1, 0)
|  |  |  |  > 1
|  |  |  > 2
|  |  |  binomial(2, 0)
|  |  |  > 1
|  |  > 3
|  |  binomial(3, 0)
|  |  > 1
|  > 4
> 10
10
```

Python shell (with @memoize)

```
binomial_memoize(5, 2)
|  binomial_memoize(4, 2)
|  |  binomial_memoize(3, 2)
|  |  |  binomial_memoize(2, 2)
|  |  |  > 1
|  |  |  binomial_memoize(2, 1)
|  |  |  |  binomial_memoize(1, 1)
|  |  |  |  > 1
|  |  |  |  binomial_memoize(1, 0)
|  |  |  |  > 1
|  |  |  > 2
|  |  > 3
|  |  binomial_memoize(3, 1)
|  |  |  binomial_memoize(2, 1)
|  |  |  > 2
|  |  |  binomial_memoize(2, 0)
|  |  |  > 1
|  |  > 3
|  > 6
|  binomial_memoize(4, 1)
|  |  binomial_memoize(3, 1)
|  |  > 3
|  |  binomial_memoize(3, 0)
|  |  > 1
|  > 4
> 10
10
```

without assigning `wrapper.__name__` the name shown would be `wrapper`

saved recursive calls when using memoization

Dynamic programming using lru_cache decorator

bionomial_lru_cache.py

```
from functools import lru_cache

@lru_cache(maxsize=None)
def binomial(n, k):
    if k==0 or k==n:
        return 1
    else:
        return binomial(n-1, k) + binomial(n-1, k-1)
```

- The decorator `@lru_cache` in the standard library `functools` supports the same as the decorator `@memoize`
- By default it at most remembers (caches) 128 previous function calls, always evicting **L**east **R**ecently **U**sed entries from its dictionary
- New in Python 3.9: `@functools.cache` idential to `lru_cache(maxsize=None)`

docs.python.org/3/library/functools.html#functools.lru_cache

Subset sum using dynamic programming

- In the subset sum problem (Exercise 13.4) we are given a number `x` and a list of numbers `L`, and want to determine if a subset of `L` has sum `x`

```
L = [3, 7, 2, 11, 13, 4, 8]    x = 22 = 7 + 11 + 4
```

- Let S(v, k) denote if it is possible to achieve value v with a subset of `L`[:k], i.e. S(v, k) = True if and only if a subset of the first k values in `L` has sum v
- S(v, k) can be computed from the following recurrence:

$$S(v, k) = \begin{cases} \text{True} & \text{if } k = 0 \text{ and } v = 0 \\ \text{False} & \text{if } k = 0 \text{ and } v \neq 0 \\ S(v, k-1) \text{ or } S(v - L[k-1], k-1) & \text{otherwise} \end{cases}$$

Subset sum using dynamic programming

subset_sum_dp.py

```python
def subset_sum(x, L):
    @memoize
    def solve(value, k):
        if k == 0:
            return value == 0
        return solve(value, k-1) or solve(value - L[k-1], k-1)
    return solve(x, len(L))
```

Python shell

```
> subset_sum(11, [2, 3, 8, 11, -1])
| True
> subset_sum(6, [2, 3, 8, 11, -1])
| False
```

Question – What is a bound on the size order of the memoization table if all values are possitive integers?

subset_sum_dp.py

```
def subset_sum(x, L):
    @memoize
    def solve(value, k):
        if k == 0:
            return value == 0
        return solve(value, k-1) or solve(value - L[k-1], k-1)
    return solve(x, len(L))
```

Python shell

```
> subset_sum(11, [2, 3, 8, 11, -1])
| True
> subset_sum(6, [2, 3, 8, 11, -1])
| False
```

a) `len(L)`

b) `sum(L)`

c) `x`

d) $2^{\texttt{len(L)}}$

e) `len(L)`

f) `len(L)*sum(L)`

g) Don't know

Subset sum using dynamic programming

subset_sum_dp.py

```
def subset_sum_solution(x, L):
    @memoize
    def solve(value, k):
        if k == 0:
            if value == 0:
                return []
            else:
                return None
        solution = solve(value, k-1)
        if solution != None:
            return solution
        solution = solve(value - L[k-1], k-1)
        if solution != None:
            return solution + [L[k-1]]
        return None
    return solve(x, len(L))
```

Python shell

```
> subset_sum_solution(11, [2, 3, 8, 11, -1])
| [3, 8]
> subset_sum_solution(6, [2, 3, 8, 11, -1])
| None
```

Knapsack problem

	Volume	Value
0	3	6
1	3	7
2	2	8
3	5	9

- Given a **knapsack** with volume **capacity C**, and set of **objects** with different **volumes and value**.
- **Objective**: Find a subset of the objects that fits in the knapsack (sum of volume ≤ capacity) and has maximal value.
- **Example**: If C = 5 and the volume and weights are given by the table, then the maximal value 15 can be achieved by the 2nd and 3rd object.
- Let V(c, k) denote the maximum value achievable by a subset of the first k objects within capacity c.

$$V(c, k) = \begin{cases} 0 & \text{if } k = 0 \\ V(c, k-1) & \text{volume}[k-1] > c \\ \max\{V(c, k-1), \text{value}[k-1] + V(c - \text{volume}[k-1], k-1)\} & \text{otherwise} \end{cases}$$

Knapsack – maximum value

knapsack.py

```
def knapsack_value(volume, value, capacity):
    @memoize
    def solve(c, k):  # solve with capacity c and objects 0..k-1
        if k == 0:  # no objects to put in knapsack
            return 0
        v = solve(c, k - 1)  # try without object k-1
        if volume[k - 1] <= c:  # try also with object k-1 if space
            v = max(v, value[k - 1] + solve(c - volume[k - 1], k - 1))
        return v

    return solve(capacity, len(volume))
```

Python shell

```
> volumes = [3, 3, 2, 5]
> values = [6, 7, 8, 9]
> knapsack_value(volumes, values, 5)
| 15
```

Knapsack – maximum value and objects

knapsack.py

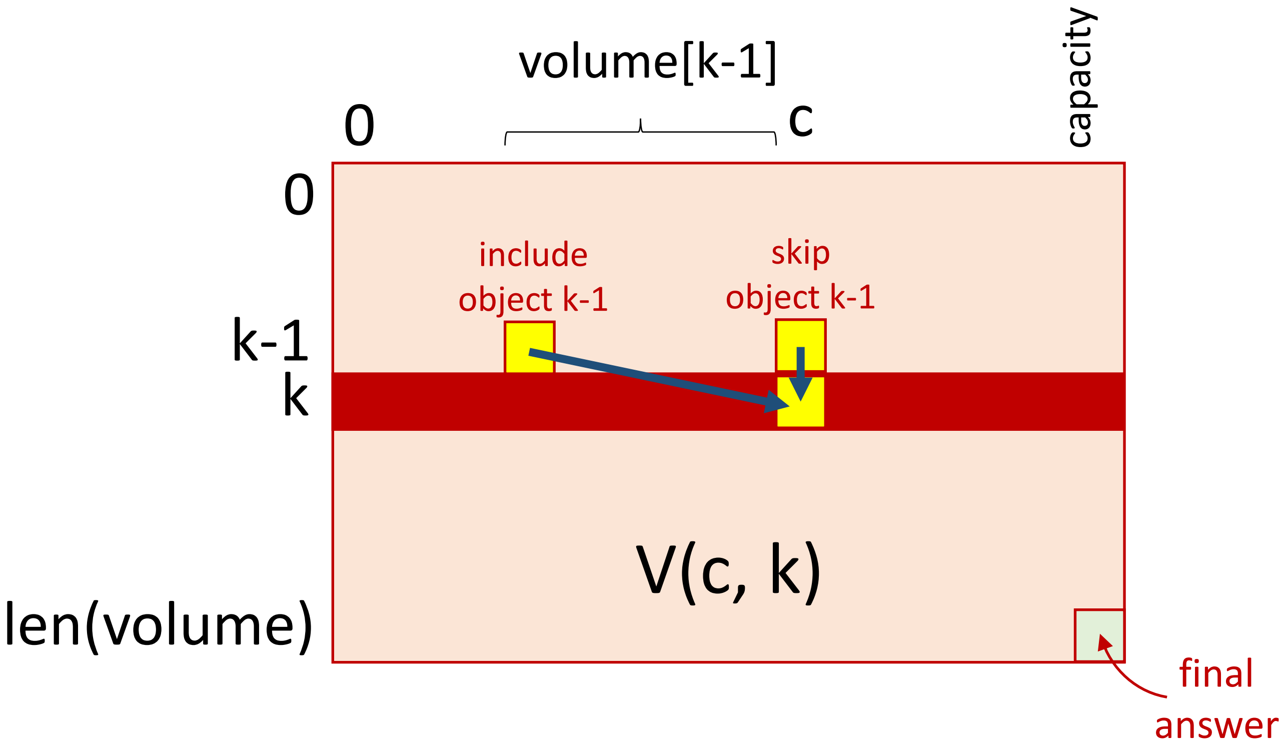
```
def knapsack(volume, value, capacity):
    @memoize
    def solve(c, k):  # solve with capacity c and objects 0..k-1
        if k == 0:  # no objects to put in knapsack
            return 0, []
        v, solution = solve(c, k-1)  # try without object k-1
        if volume[k - 1] <= c:  # try also with object k-1 if space
            v2, sol2 = solve(c - volume[k - 1], k - 1)
            v2 = v2 + value[k - 1]
            if v2 > v:
                v = v2
                solution = sol2 + [k - 1]
        return v, solution

    return solve(capacity, len(volume))
```

Python shell

```
> volumes = [3, 3, 2, 5]
> values = [6, 7, 8, 9]
> knapsack(volumes, values, 5)
| (15, [1, 2])
```

Knapsack - Table

$$V(c, k) = \begin{cases} 0 & \text{if } k = 0 \\ V(c, k-1) & \text{value}[k-1] > c \\ \max\{V(c, k-1), \text{value}[k-1] + V(c - \text{volume}[k-1], k-1)\} & \text{otherwise} \end{cases}$$



- systematic fill out table
- only need to remember two rows

Knapsack – Systematic table fill out

knapsack_systematic.py

```
def knapsack(volume, value, capacity):
  ① solutions = [(0, [])] * (capacity + 1)
  ② for obj in range(len(volume)): ⑤
         for c in reversed(range(volume[obj], capacity + 1)): ④
             prev_v, prev_solution = solutions[c - volume[obj]]
             v = value[obj] + prev_v
             if solutions[c][0] < v:
               ③ solutions[c] = v, prev_solution + [obj]

     return solutions[capacity]
```

Python shell

```
> volumes = [3, 3, 2, 5]
> values = [6, 7, 8, 9]
> knapsack(volumes, values, 5)
| (15, [1, 2])
```

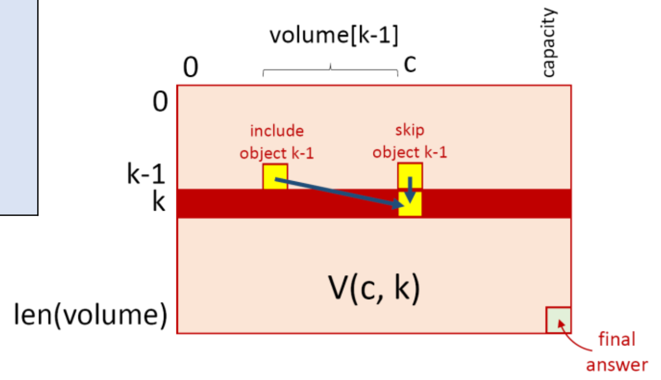
① base case k = 0

② consider each object

③ `solutions[c:]` current row
`solutions[:c]` previous row

④ compute next row right-to-left

⑤ `solutions[:volume[obj]]`
unchanged from previous row



Summary

- Dynamic programming is a general approach for recursive problems where one tries to avoid recomputing the same expresions repeatedly

- **Solution 1: Memoization**
  - add dictionary to function to remember previous results
  - decorate with a @memoize decorator
- **Solution 2: Systematic table fill out**
  - can need to compute more values than when using memoization
  - can discard results not needed any longer (reduced memory usage)

Coding competitions and online judges

If you like to practice your coding skills, there are many online "judges" with numerous exercises and where you can upload and test your solutions.

- Project Euler
- Kattis
- Google Code Jam
- CodeForces
- Topcoder

See cs.au.dk/~gerth/code/

Google Code Jam

codingcompetitions.withgoogle.com/codejam

code jam

print "hello, world!"

- **Coding competition**
- **Qualification round 2020** (April 4, 01:00 – April 5, 04:00)
- In 2019 there was 30.000 participants for the qualification round

Scoreboard 2017

Rank	Contestant	Score	Penalty	A. Oversized Pancake Flipper 5pt	A. 10pt	B. Tidy Numbers 5pt	B. 15pt	C. Bathroom Stalls 5pt	C. 10pt	C. 15pt	D. Fashion Show 10pt	D. 25pt
1	FatalEagle	100	55:12	4:02	4:29	9:02	9:25	22:30	22:50	23:13	54:48	55:12
2	ACMonster	100	57:32	3:41	4:02	11:37	11:58	25:03	25:27	26:01	57:08	57:32
3	y0105w49	100	58:00	8:28	8:58	16:20	16:47	31:34 1 wrong try	32:03	32:36	53:27	54:00
621	MathCrusader	75	11:52:45	1:52:49	1:53:20	2:11:37 1 wrong try	2:12:05	2:18:48	2:19:18	2:19:48	11:24:45 6 wrong tries	-- 1 wrong try
622	Siddhant22	75	12:07:37	2:15:58 1 wrong try	2:18:01	3:05:10	3:06:39	4:42:18	4:43:32	4:46:14	11:59:37 1 wrong try	Time expired
623	**daimi89** Me	**75**	**12:18:31**	6:21:53	6:22:58	7:15:07	7:15:54	7:52:33	7:53:13	7:54:04	12:14:31 1 wrong try	--
624	plamenko	75	12:19:28	11:20	11:56	27:38	28:23	42:34	43:17	43:55	12:11:28 2 wrong tries	-- 1 wrong try

Google Code Jam - Qualification Round 2017
Problem A: Oversized Pancake Flipper (description)

- N pancakes each with exactly one happy chocolate side
- K-flipper that can flip K consecutive pancakes
- Problem: Find minimim number of flips to make all pancakes happy, if possible