I/O-comparison trees

[Arge, Knudsen, Larsen, 93]

Result

Goal:

A general reduction theorem:

Lower bound on comparisons to solve a problem

$\Downarrow$

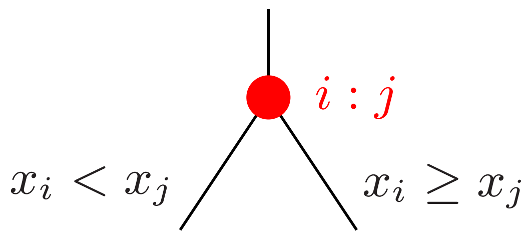
Lower bound on I/Os to solve the problem

Method:

Extend the notion of comparison trees.

Standard Comparison Trees

- Binary trees.
- Internal node labelled with pairs of elements, represents comparisons.
- Edges labelled with one of the possible outcomes of the comparison above.
- Leaves labelled with one possible answer to problem ("Yes/No" for decision problems, a permutation for construction problems, an element for search problems)



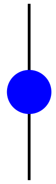
Tree **solves** a problem

$\Downarrow$

$\forall$ leaves l: $\forall$ input x ending in l: label of l is correct for x.

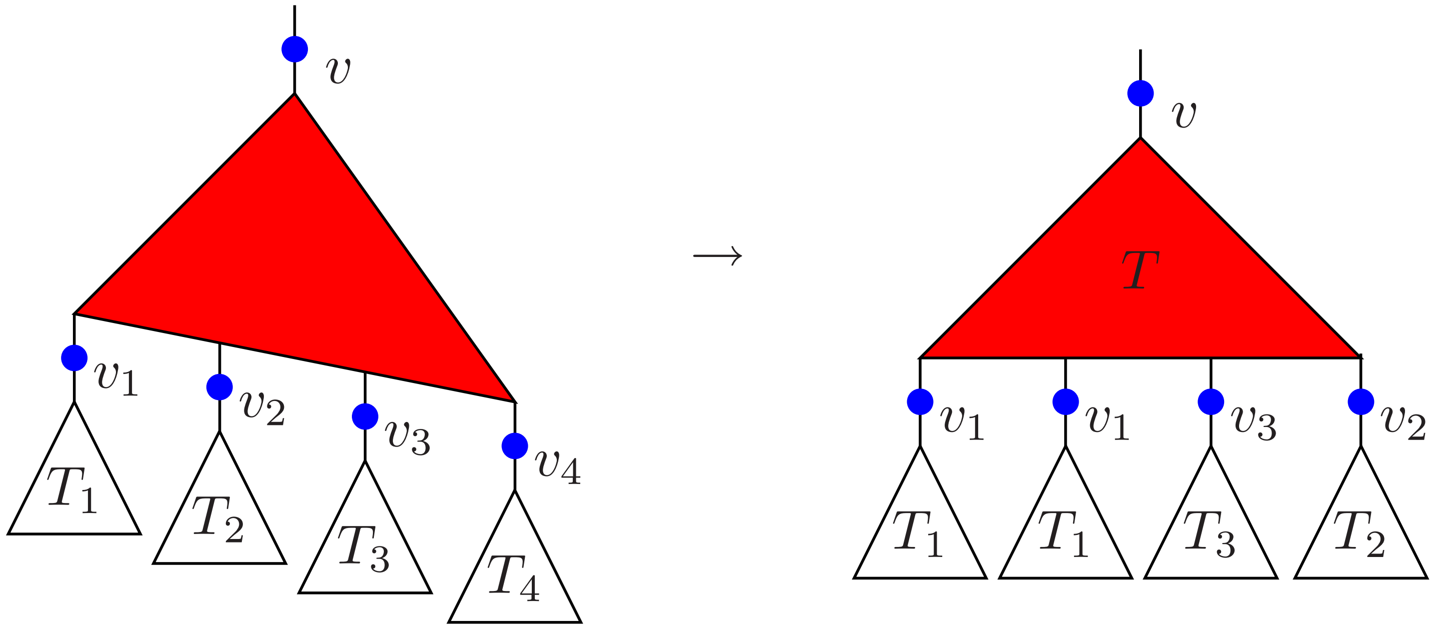
I/O Comparison Trees

- Add unary **I/O-nodes** to comparison trees.
- I/O node labelled with position in memory of all elements before and after I/O.
- Root and leaves: I/O-nodes.
- Comparison nodes may only compare nodes in RAM (given by label of lowest ancestor which is an I/O-node).



Compression

Compress comparison-only subtrees:



T: minimal height comparison tree to sort contents of RAM at v.

Reduction

Compress entire tree by compressing all comparison-only subtrees in top-down order:

$$T_1 \to T_2$$

By induction on number of I/O-nodes on path: an input x will pass exactly the same I/O-nodes (same number of nodes having the same labels) in T_1 and T_2.

Corollary: x ends up in leaf with same label in T_1 and T_2.

Finally, remove all I/O-nodes from T_2:

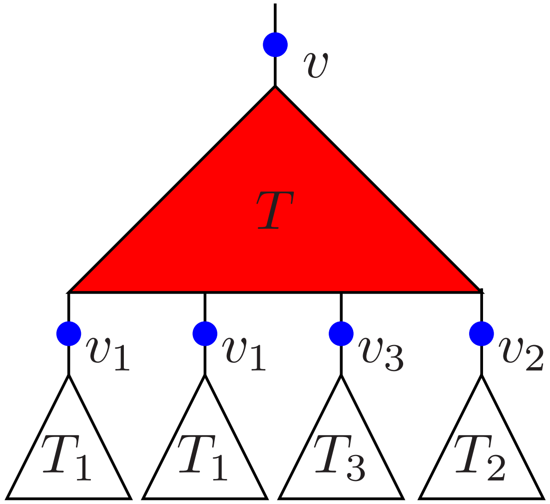
$$T_2 \to T_3$$

Now T_3 is standard comparison tree solving same problem.

Height of T

Theorem: Comparison complexity of sorting n elements is $\Theta(n \log n)$.

Theorem: Comparison complexity of merging two sorted lists of lengths n and m is $\Theta(m(\log(n/m) + 1))$, assuming $n \geq m$.



Type of I/O at v	Height of T at most
Untouched	$B \log B + B \log((M - B)/B)$
Touched	$B \log((M - B)/B)$

At most N/B untouched blocks.

Reduction Analysis

$\forall$ inputs x:

$$
\begin{aligned}
|\text{path in } T_3| &= |\text{sti i } T_2| - [\text{I/Os in } T_2] \\
&\leq [\text{I/Os in } T_2] \cdot (B \log(M/B) - 1 - 1) + (N/B) B \log B \\
&\leq [\text{I/Os in } T_2] \cdot B \log(M/B) + (N/B) B \log B \\
&\leq [\text{I/Os in } T_1] \cdot B \log(M/B) + N \log B
\end{aligned}
$$

$\exists$ comparison lower bound $L \Rightarrow L \leq |\text{path in } T_3|$

$$\frac{L - N \log B}{B \log(M/B)} \leq \text{I/Os in } T_1$$

Examples

$$\frac{L - N\log B}{B\log(M/B)} \leq \text{I/Os in } T_1$$

Problem	L	I/O Lower Bound
Sorting	$N\log N$	$(N/B)\log_{M/B}(N/B)$
Set equality	$N\log N$	do.
Set inclusion	$N\log N$	do.
Set disjointness	$N\log N$	do.

Multiset sorting, duplicate removal, mode finding: see paper.