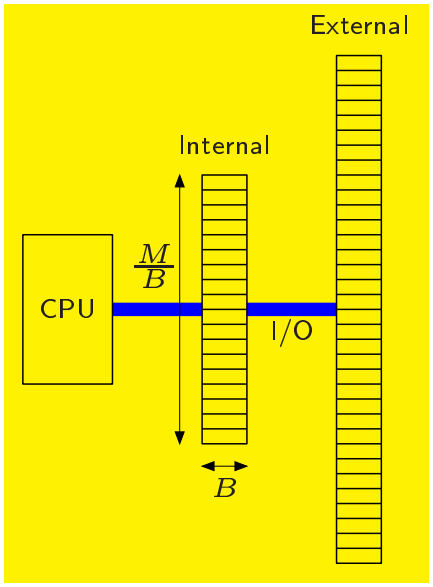
# Cache-Oblivious Algorithms

[FLPR99] M. Frigo, C. E. Leiserson, H. Prokop, S. Ramachandran. *Cache-Oblivious Algorithms*. In Proc. 40th Annual Symposium on Foundations of Computer Science (FOCS '99), 1999.

# Models

## I/O Model [Aggarwal & Vitter]

- Algorithms are resource-aware since variables depend on $M$ and $B$
- Algortihms have control over I/O transfers




## Ideal-Cache Model [Frigo et al.]

- Two-level memory hierarchy
- Ideal data cache of size $M$
- Cache lines of size $B$
- Fully associative cache
  (cache lines can be stored anywhere in cache)
- Optimal off-line cache replacement strategy
  (replace the cache line whose next access is furthest in the future)
- Tall cache assumption : $M = \Omega(B^2)$

# Cache-Oblivious Algorithms

## Cache-aware algorithms

- Parameters can be tuned to optimize cache complexity
- Examples
  - $\Theta(\frac{M}{B})$-way merging
  - $\Theta(\frac{M}{B})$-way MergeSort

## Cache-oblivious algorithms

- No variable depends on hardware parameters
- Examples
  - Scanning
  - Classic linear time selection
  - 2-way merging
  - 2-way MergeSort $O(\frac{N}{B}\log_2 N)$ I/Os $\Rightarrow$ not optimal !

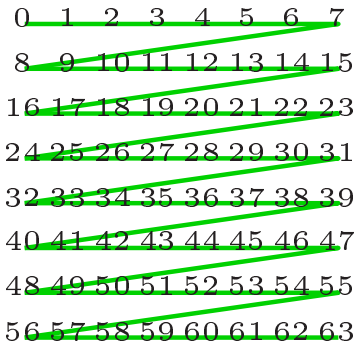
# Matrix Multiplication

## Problem

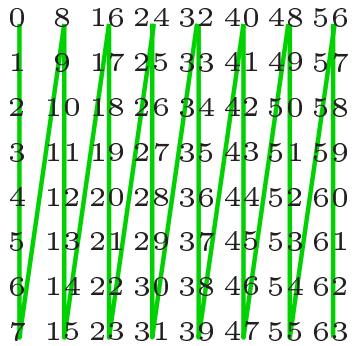
Multiply two $N \times N$ matrices A and B

$$C = A \cdot B\ , \quad c_{ij} = \sum_{k=1..N} a_{ik} \cdot b_{kj}$$

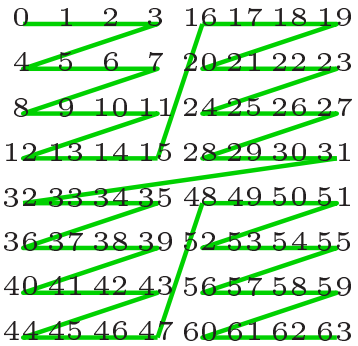
## Layout of matrices



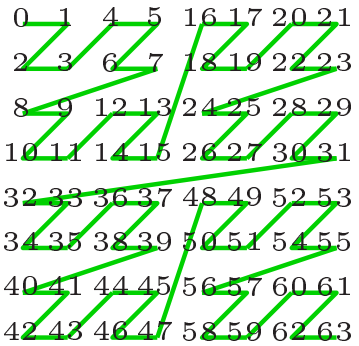

Row major




Column major




$4 \times 4$-blocked




Bit interleaved

# Matrix Multiplication

## Algorithm 1: Nested loops

- Row major
- Reading a column of $B$ uses $O(N)$ I/Os
- Total $O(N^3)$ I/Os

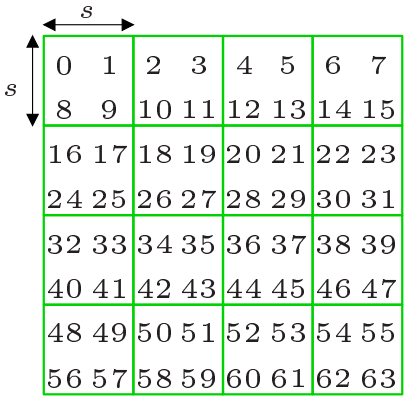
```
for i = 1 to N
  for j = 1 to N
    c_ij = 0
    for k = 1 to N
      c_ij = c_ij + a_ik · b_kj
```

## Algorithm 2: Blocked algorithm (cache-aware)

- Partition $A$ and $B$ into blocks of size $s \times s$ where $s = \Theta(\sqrt{M})$
- Apply Algorithm 1 to the $\frac{N}{s} \times \frac{N}{s}$ matrices where elements are $s \times s$ matrices
- $s \times s$-blocked or ( row major and $M = \Omega(B^2)$ )

$$O\left(\left(\frac{N}{s}\right)^3 \cdot \frac{s^2}{B}\right) = O\left(\frac{N^3}{s \cdot B}\right) = O\left(\frac{N^3}{B\sqrt{M}}\right) \text{ I/Os}$$

- Optimal due to lower bound by [Hong & Kung]



# Matrix Multiplication

## Algorithm 3: Recursive algorithm (cache-oblivious)

- 8 recursive $\frac{N}{2} \times \frac{N}{2}$ matrix multiplications + 4 $\frac{N}{2} \times \frac{N}{2}$ matrix sums

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$$

- #I/Os if bit interleaved or ( row major and $M = \Omega(B^2)$ )

$$T(N) \leq \begin{cases} O(\frac{N^2}{B}) & \text{if } N \leq \varepsilon\sqrt{M} \\ 8 \cdot T\left(\frac{N}{2}\right) + O\left(\frac{N^2}{B}\right) & \text{otherwise} \end{cases}$$

$$T(N) \leq O\left(\frac{N^3}{B\sqrt{M}}\right)$$

- Optimal due to lower bound by [Hong & Kung]

[FLPR99] General case: $A$ an $N_1 \times N_2$ matrix, $B$ an $N_2 \times N_3$ matrix

# Matrix Multiplication

## Algorithm 4: Strassen's algorithm (cache-oblivious)

- Exploits subtractions
- 7 recursive $\frac{N}{2} \times \frac{N}{2}$ matrix multiplications + $O(1)$ matrix sums

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

$$\begin{aligned}
m_1 &:= (a_{21} + a_{22} - a_{11})(b_{22} - b_{12} + b_{11}) \\
m_2 &:= a_{11} b_{11} \\
m_3 &:= a_{12} b_{21} \\
m_4 &:= (a_{11} - a_{21})(b_{22} - b_{12}) \\
m_5 &:= (a_{21} + a_{22})(b_{12} - b_{11}) \\
m_6 &:= (a_{12} - a_{21} + a_{11} - a_{22}) b_{22} \\
m_7 &:= a_{22}(b_{11} + b_{22} - b_{12} - b_{21})
\end{aligned}$$

$$\begin{aligned}
c_{11} &:= m_2 + m_3 \\
c_{12} &:= m_1 + m_2 + m_5 + m_6 \\
c_{21} &:= m_1 + m_2 + m_4 - m_7 \\
c_{22} &:= m_1 + m_2 + m_4 + m_5
\end{aligned}$$

- #I/Os if bit interleaved or ( row major and $M = \Omega(B^2)$ )

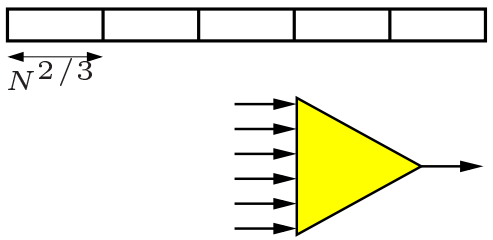
$$T(N) \leq \begin{cases} O(\frac{N^2}{B}) & \text{if } N \leq \varepsilon\sqrt{M} \\ 7 \cdot T\left(\frac{N}{2}\right) + O\left(\frac{N^2}{B}\right) & \text{otherwise} \end{cases}$$

$$T(N) \leq O\left(\frac{N^{\log_2 7}}{B\sqrt{M}}\right) \qquad \log_2 7 \approx 2.81$$

# Merge-Sorting : Funnelsort

## Funnelsort

1. Partition input into $N^{1/3}$ peaces of size $N^{2/3}$
2. Sort each peace recursively
3. Merge sorted peaces using an $N^{1/3}$-merger




## $k$-merger

A $k$-merger takes $k$ sorted input sequences

Each invocation of a $k$-merger outputs the next $k^3$ elements

## Theorem

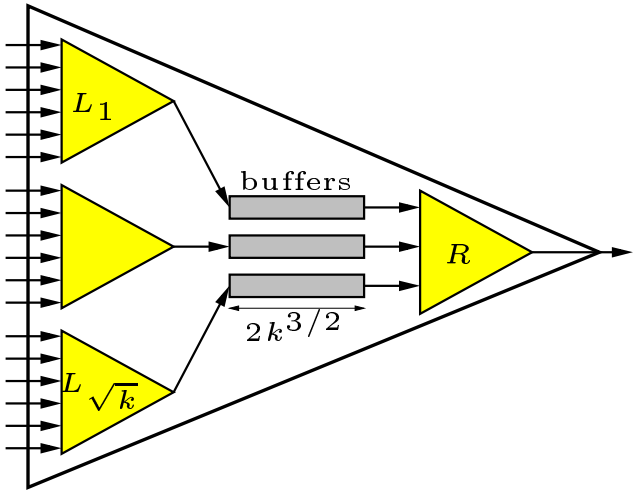
If an invocation of a $k$-merger uses $O(k + \frac{k^3}{B} \log_M k^3)$ I/Os then funnelsort uses $O(\frac{N}{B} \log_M N)$ I/Os

**Proof**

$$T(N) = \begin{cases} \frac{N}{B} & \text{if } N \leq M \\ N^{1/3} \cdot T(N^{2/3}) + \frac{N}{B} \log_M N & \text{otherwise} \end{cases}$$

□

# $k$-merger




- $\sqrt{k}+1$ recursive $\sqrt{k}$-mergers
- $\sqrt{k}$ buffers (FIFO) of size $2k^{3/2}$
  Implemented as cyclic queues in arrays
  (efficient scanning of input streams)
- 2-merger outputs 8 elements

## Invocation of a $k$-merger

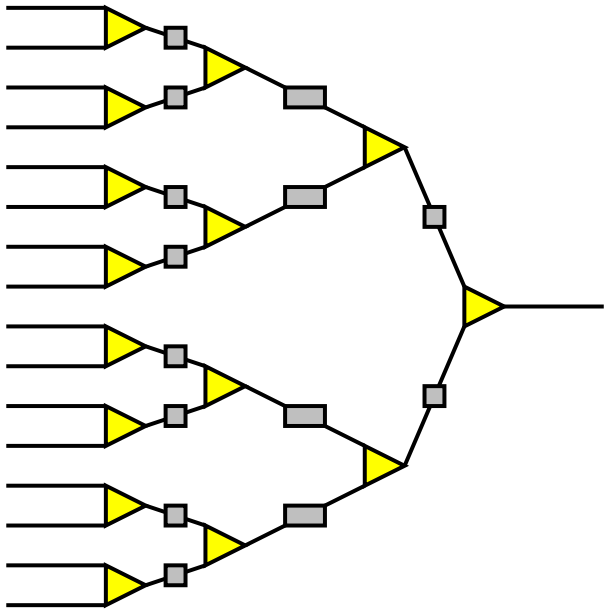
- $k^{3/2}$ times invoke the $\sqrt{k}$-merger $R$
- Before invoking $R$ invoke all $L_i$ where the buffer contains less than $k^{3/2}$ elements (and the input streams are not exhausted)

**Lemma** Can be laid out in $O(k^2)$ contigious memory locations

**Proof** $S(k) = (\sqrt{k}+1) \cdot S(\sqrt{k}) + O(2k^{3/2} \cdot \sqrt{k})$ □

# $k$-merger

## Example : 16-merger



$k$-merger $\equiv$ a scheduling of 2-mergers

## Memory accesses

**2-merger** : Reads/writes sequentially input and output

$k$-**merger** : $k^{3/2} \cdot \sqrt{k} = k^2$ reads to check if input buffers $\geq k^{3/2}$

# $k$-merger

**Lemma** A $k$-merger makes $O(k + \frac{k^3}{B} \log_M k^3)$ I/Os per invocation

**Case I:** $k \leq \varepsilon\sqrt{M}$

Space $\underbrace{k \cdot B}_{\text{input}} + \underbrace{B}_{\text{output}} + \underbrace{O(k^2)}_{\text{buffers}} = O(M)$ since $M = \Omega(B^2)$

i.e. computation can be done in internal memory

I/Os $O(\underbrace{k + k^3/B}_{\text{input/output}} + \underbrace{k^2/B}_{\text{touching buffers}}) = O(k + k^3/B)$

**Case II:** $k > \varepsilon\sqrt{M}$

At some level in the recursion $\sqrt{\varepsilon\sqrt{M}} < \bar{k} \leq \varepsilon\sqrt{M}$

$$\bar{k}^2 \geq \varepsilon\sqrt{M} = \Omega(B) \quad \Rightarrow \quad \bar{k}^3/B = \Omega(\bar{k})$$

By Case I a $\bar{k}$-merger uses $O(\bar{k}^3/B) = O(\frac{\text{output}}{B})$ I/Os

# $k$-merger

## Case II: $k > \varepsilon\sqrt{M}$ (continued)

I/Os used for merging $k^3$ elements in a $k$-merger

- In $\bar{k}$-mergers : $O(\frac{k^3}{B} \cdot \log_{\bar{k}} k)$ I/Os
- Checking buffer sizes between two $\bar{k}$-mergers

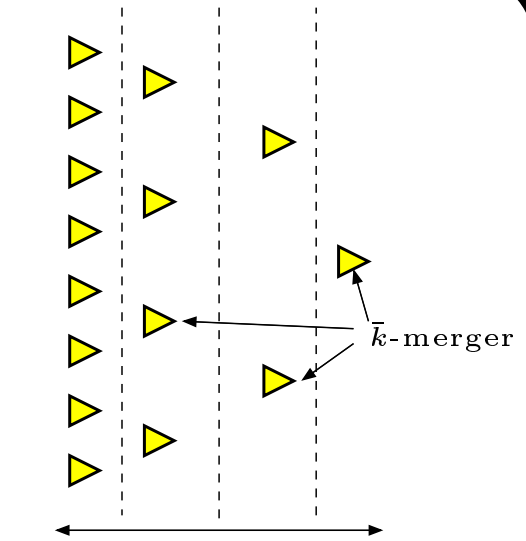
  Invoking a $\hat{k}$-merger, $\hat{k} > \bar{k}$

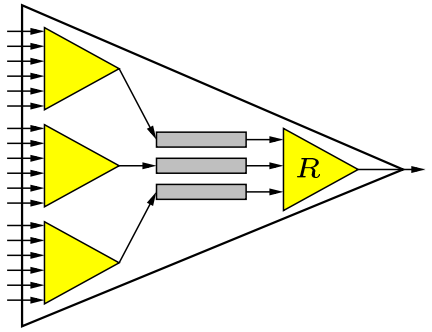
  - $\hat{k}^2$ reads to check buffer sizes
  - $\hat{k}^3$ elements propagated from the buffers
  - I/Os per element propagated

$$\frac{\hat{k}^2}{\hat{k}^3} = \frac{1}{\hat{k}} = O(\frac{1}{\sqrt{M}}) = O(\frac{1}{B})$$

Total #I/Os for buffer checks is

$$O(\frac{1}{B} \cdot k^3 \log_{\bar{k}} k) = O(\frac{k^3}{B} \log_M k)$$




□

# Justification of the ideal-cache model

## Optimal replacement

LRU + 2 $\times$ cache size $\Rightarrow$ at most 2 $\times$ cache misses [Sleator & Tarjan]

## Corollary

$T_{M,B}(N) = O(T_{2M,B}(N)) \Rightarrow$ #cache misses using LRU is $O(T_{M,B}(N))$

## Two memory levels

Optimal cache-oblivious algorithm satisfying $T_{M,B}(N) = O(T_{2M,B}(N))$
$\Rightarrow$ optimal #cache misses on each level of a multilevel cache using LRU

## Fully associativity cache

Simulation of LRU

- Direct mapped cache
- Explicit memory management
- Dictionary (2-universal hash functions) of cache lines in memory
- Expected $O(1)$ access time to a cache line in memory