Alcom-FT Technical Report Series
ALCOMFT-TR-01-86

Scalable Sparse Topologies with Small Spectrum *

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Abstract. One of the fundamental properties of a graph is the number of distinct eigenvalues of its adjacency or Laplacian matrix. Determining this number is of theoretical interest and also of practical impact. Graphs with small spectra exhibit many symmetry properties and are well suited as interconnection topologies. Especially load balancing can be done on such interconnection topologies in a small number of steps. In this paper we are interested in graphs with maximal degree $O(\log n)$, where n is the number of vertices, and with a small number of distinct eigenvalues. Our goal is to find scalable families of such graphs with polylogarithmic spectrum in the number of vertices. We present also the eigenvalues of the Butterfly graph.

1 Introduction

Spectral methods in graph theory have received great attention since their introduction and have proved to be a valuable tool for the theoretical and applied graph theory [7, 3]. A (Laplacian or adjacency) matrix is associated to each graph. The set of the eigenvalues of this matrix is called the (Laplacian or adjacency) spectrum of the graph; it is one of the most important algebraic invariants of a graph. Although in general a graph is not characterized uniquely by its spectrum, there is a strong connection between the eigenvalues and many structural properties of the graph (diameter, bisection width, expansion etc). See [5] for a selection of results in this area.

An important parameter connected to a spectrum of a graph is its size, i.e. the number of distinct eigenvalues of the adjacency (or Laplacian) matrix of the graph. This value is correlated to the symmetry properties of it: the only graph having two distinct eigenvalues is the complete graph and its automorphism group is as rich as possible – the symmetric group. Graphs having three distinct eigenvalues are called strongly regular; their diameter is 2 and they posses many interesting properties [16, 3]. Another well studied class of highly symmetric graphs are distance-regular graphs [4]. The size of their spectrum is $1 + diam(G)$ which matches a lower bound for all graphs.

In the past there have been written several papers about well structured graphs. Consider for example the hypercube $Q(d)$ as a vertex and edge symmetric graph. It has 2^d vertices and a diameter resp. vertex degree of d. The hypercube has $d + 1$ distinct

* This work was partially supported by the German Research Association (DFG) within the SFB 376 “Massive Parallelität: Algorithmen, Entwurfsmethoden, Anwendungen” and by the IST Programme of the EU under contract number IST-1999-14186 (ALCOM-FT).

** The research was done while the second author was visiting the Department of Mathematics and Computer Science at the University of Paderborn

eigenvalues and a large application as an interconnection topology. Other graphs as cliques, complete bipartite graphs or the star have only 2 or 3 distinct eigenvalues, but because of their high density are ill-suited as interconnection topologies.

There exists some graphs with an even better relation between number of vertices, vertex degree, diameter and number of eigenvalues than the hypercube. One of them is the Petersen graph, which has 10 vertices, a vertex degree of 3, diameter 2 and 3 different eigenvalues. Another one is the $Cage(6,6)$, which has 62 vertices, vertex degree 6, a diameter of 2 and only 3 distinct eigenvalues. A family of graphs with a very good behavior is the family of the star graphs [2, 1]. The star graph of order d has $d!$ vertices, a vertex degree of $d-1$, diameter $\frac{3}{2}(d-1)$ and using a previous work of Flatto, Odlyzko and Wales it turns out that it has only $2d-1$ distinct eigenvalues [12].

In this paper we focus our attention on constructing scalable families of sparse graphs (maximal vertex degree $O(\log n)$ where n is the number of vertices) with small spectra. We use the term scalable to denote a family of graphs, which contains for each natural n an n-vertex graph. Our motivation for studying this question comes from the area of load balancing in distributed systems. Let there be given an arbitrary, undirected, connected graph $G=(V,E)$ in which node $v \in V$ contains a load of $w(v)$. The goal is to determine a schedule to move load across edges so that finally the load on each node will be the same. In each step load can be moved from any node to its neighbors. Communication between non-adjacent vertices is not allowed. This problem describes load balancing in synchronous distributed processor networks and parallel machines when we associate a node with a processor, an edge with a communication link and the load with identical, independent tasks [6, 18].

Load balancing algorithms are typically based on a fixed topology which defines the load balancing partners in the system. Consider for example a bus system where each processor can communicate with any other processor in the network. To avoid high communication costs, we allow any processor to communicate only with a small number of other nodes in the system. Then we can define a topology, which has a small vertex degree and supports fast load balancing on the network. See also [8] for a practical point of view of this problem.

Now the load balancing process can be split into two phases, the flow computation phase, which computes the network flow, and the migration phase, which migrates the load items according to the computed flow. Algorithms for the flow computation phase have been extensively studied. Many of them are local iterative schemes based on diffusion or dimension exchange [10, 11, 13–15, 18]. The diffusion algorithms studied in the above mentioned papers calculate an l_2-optimal flow. We are interested in topologies, for which the optimal scheme OPT [11] (see also the next section) has a small number of iteration steps (polylogarithmic in the number of vertices). Applying the optimal scheme we need only $m-1$ iterations where m is the number of distinct eigenvalues of the Laplacian of the graph. In any iteration step a node has to communicate with all of its neighbors, so the total cost of the load balancing algorithm depends on the number of the distinct eigenvalues of the graph and on its vertex degree. The number of steps is in fact the product of both. Therefore we are interested in topologies with a small product of the vertex degree and of the number of distinct eigenvalues.

The paper is organized as follows. In section 2 we present the definitions and lemmas used in this paper to compute the eigenvalues of the graphs constructed below. In section 3 we compute the spectrum of the Butterfly graph and propose scalable families of trees of constant degree whose spectrum consists of $O(\log^2 n)$ different eigenvalues. Since the tree topology is not well suited for our application we present a scalable family of well connected graphs with at most $O(\log^3 n)$ distinct eigenvalues and a vertex degree of $O(\log n)$. In the last section we improve the previous results for the case of the adjacency spectrum using another technique to obtain $O(\log^2 n)$ distinct eigenvalues, where the vertex-degree still remains $O(\log(n))$.

Concerning the product of the vertex degree and the number of distinct eigenvalues, the star graphs are the best graphs we know. In their case this product is $O((\frac{\log n}{\log\log n})^2)$ where n is the number of vertices. We found scalable families of graphs with a good behavior, but we did not reach this bound. We also do not know, if this bound is optimal. The only lower bound is $\Omega(\log n)$. So there are a lot of open problems which are left to be solved in this important field.

2 Definitions and Lemmas

In this paper we are interested in scalable families of graphs. A family of graphs $\mathcal{G}$ is called scalable, if for each $n \in \mathbb{N}$ there is an n-vertex graph $G \in \mathcal{G}$. The identity matrix will be denoted $\mathbf{I}_n \in \mathbb{R}^{n\times n}$. Symbols $\mathbf{J}_{m,n}$, $\mathbf{0}_{m,n}$ denote $m \times n$ matrices containing all ones and all zeros, respectively. The spectrum of a matrix $\boldsymbol{A}$ is the set of its eigenvalues: $Sp(\boldsymbol{A}) = \{\lambda \mid \exists \boldsymbol{x} \ : \ \boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}\}$

The operation "$\otimes$" denotes the *Kronecker product*: for the matrices $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{B} \in \mathbb{R}^{p\times q}$ the matrix $\boldsymbol{A} \otimes \boldsymbol{B} \in \mathbb{R}^{mp\times nq}$ is the matrix obtained from $\boldsymbol{A}$ by replacing every element a_{ij} by the block $a_{ij}\boldsymbol{B}$.

Consider a (weighted) digraph $G = (V, E)$ with $w(e)$ being the weight of an edge e. The *adjacency matrix* of G is the matrix $\boldsymbol{A}_G = (a_{ij})_{1\le i,j\le |V|}$ where $a_{ij} = w(e_{ij})$ if an edge e_{ij} leads from a vertex v_i to a vertex v_j and $a_{ij} = 0$ otherwise (a_{ii} is the weight of a self-loop in v_i). The *Laplacian matrix* of G is the matrix $\boldsymbol{\Delta}_G = \boldsymbol{D} - \boldsymbol{A}_G$ where $\boldsymbol{D} = (d_{ij})$ is a diagonal matrix with entries $d_{ii} = \sum_j a_{ij}$. The spectrum of adjacency and Laplacian matrix of a graph G will be denoted $Sp_A(G)$ and $Sp_\Delta(G)$ and called adjacency and Laplacian spectrum of G, respectively. Note that for d-regular graphs the adjacency and Laplacian spectrum are equivalent. The Laplacian spectrum of a d-regular graph consists of all values $\lambda_\Delta = d - \lambda_A$, where λ_A is an eigenvalue of the adjacency matrix and d is the vertex degree of the graph.

We are looking in this paper for graphs where the optimal scheme OPT has a small number of iteration steps. The optimal scheme is defined as follows. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the m nonzero distinct eigenvalues of the Laplacian of the graph. Now, in the t-th iteration step each vertex v_i sends a load of $\frac{1}{\lambda_t} w_i^t$ to its neighbors, where w_i^t is the load of vertex v_i after the iteration step $t-1$. So in the t-th iteration step, the load of the vertex v has the form

$$w^{t+1}(v) = w^t(v) - \sum_{\{v,u\}\in E} \frac{1}{\lambda_t}(w^t(v) - w^t(u))$$

After m steps, the load of the network will be totally balanced (see [11]). Note that this implies that the diameter of the graph is less than or equal to $m-1$ (see also [7]).

Following there is a well known lower bound on the size of the spectrum:

Lemma 1. *The number of distinct eigenvalues of the adjacency and Laplacian matrix of an undirected connected graph G with n vertices and maximal degree d is $\Omega\left(\frac{\log n}{\log d}\right)$.*

Proof. We have seen above that for the diameter of G it holds $diam(G) \leq |Sp_{\Delta}(G)| - 1$ (e.g. [7]). By an argument known as Moore's bound: a graph with maximal degree d and diameter $diam(G)$ can have at most $1 + d + d(d-1) + \cdots + d(d-1)^{diam(G)-1}$ vertices, the lemma follows. For the case of the adjacency spectrum a similar argument holds ([7]). □

In our approach to the construction of graphs with small spectra we shall use the following lemmas. The intuition behind is that having a graph G with the corresponding (adjacency or Laplacian) matrix $\boldsymbol{A}$, we transform $\boldsymbol{A}$ using a suitable non-singular matrix $\boldsymbol{X}$ into a block-diagonal form. The spectrum of $\boldsymbol{A}$ is given by the union of the spectra of the particular block matrices. It is often convenient to view these block components as matrices of some simpler graphs.

Lemma 2. *Let $n = p \cdot m + r$, $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ be a matrix of the form*

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{C} & \mathbf{J}_{1,p} \otimes \boldsymbol{S} \\ \mathbf{J}_{p,1} \otimes \boldsymbol{R} & \mathbf{I}_p \otimes \boldsymbol{B} + (\mathbf{J}_{p,p} - \mathbf{I}_p) \otimes \boldsymbol{X} \end{pmatrix}$$

where $\boldsymbol{C} \in \mathbb{R}^{r \times r}$, $\boldsymbol{S} \in \mathbb{R}^{r \times m}$, $\boldsymbol{R} \in \mathbb{R}^{m \times r}$ and $\boldsymbol{B}, \boldsymbol{X} \in \mathbb{R}^{m,m}$. Then the spectrum of $\boldsymbol{A}$ can be written as the union

$$Sp(\boldsymbol{A}) = Sp(\boldsymbol{B} - \boldsymbol{X}) \cup Sp\begin{pmatrix} \boldsymbol{C} & \sqrt{p} \cdot \boldsymbol{S} \\ \sqrt{p} \cdot \boldsymbol{R} & \boldsymbol{B} + (p-1)\boldsymbol{X} \end{pmatrix} \tag{1}$$

Proof. Consider the matrices

$$\boldsymbol{W} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \boldsymbol{U} \otimes \mathbf{I}_m \end{pmatrix}, \boldsymbol{U} = \frac{1}{\sqrt{p}} \begin{pmatrix} \mathbf{I}_{p-1} & -\mathbf{J}_{p-1,1} \\ \mathbf{J}_{1,p-1} & 1 \end{pmatrix} \tag{2}$$

and

$$\boldsymbol{U}^{-1} = \frac{1}{\sqrt{p}} \begin{pmatrix} (p-1) \cdot \mathbf{I}_{p-1} - \mathbf{J}_{p-1,p-1} & \mathbf{J}_{p-1,1} \\ -\mathbf{J}_{1,p-1} & 1 \end{pmatrix} \tag{3}$$

Using the transformation $Sp(\boldsymbol{A}) = Sp(\boldsymbol{W}\boldsymbol{A}\boldsymbol{W}^{-1})$ we get

$$Sp(\boldsymbol{A}) = Sp\begin{pmatrix} \boldsymbol{C} & \mathbf{0} & \sqrt{p} \cdot \boldsymbol{S} \\ \mathbf{0} & \mathbf{I}_{p-1} \otimes (\boldsymbol{B} - \boldsymbol{X}) & \mathbf{0} \\ \sqrt{p} \cdot \boldsymbol{R} & \mathbf{0} & \boldsymbol{B} + (p-1)\boldsymbol{X} \end{pmatrix}$$

The lemma follows by interchanging the second and the third row and column block. □

As an example we present the case $p = 2$. Taking into consideration, that in this work we consider only symmetric matrices, $\boldsymbol{S} = R^T$ holds and the matrix A has the form

$$\begin{pmatrix} \boldsymbol{C} & \boldsymbol{R}^T & \boldsymbol{R}^T \\ \boldsymbol{R} & \boldsymbol{B} & \boldsymbol{X} \\ \boldsymbol{R} & \boldsymbol{X} & \boldsymbol{B} \end{pmatrix} \text{ which is transformed by lemma 2 to } \begin{pmatrix} \boldsymbol{C} & \sqrt{2} \cdot \boldsymbol{R}^T & \mathbf{0} \\ \sqrt{2} \cdot \boldsymbol{R} & \boldsymbol{B} + \boldsymbol{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\boldsymbol{B} - \boldsymbol{X}) \end{pmatrix}$$

The matrix A above can be viewed as the adjacency matrix of a graph, which is constructed from a core (with the adjacency matrix C) and 2 copies of a graph with the adjacency matrix B. The subgraph B appears 2 times in the graph and therefore some eigenvalues appear also more than once.

From now on we will not distinguish between the notation of a matrix and an edge weighted graph. The vertices of A are connected to some vertices of both copies of B and these edges are represented by the matrix R. Now the spectrum of this is the union of the spectrum of two other graphs. The first is constructed from C and one copy of B, where the matrix $\sqrt{2} \cdot R$ represents the edges between C and $B + X$. The second graph will be $B - X$.

In the following sections we consider $\mathbf{I}$ or $\mathbf{0}$ as the matrix $\boldsymbol{X}$. We define Q as the smallest matrix with the property: $\boldsymbol{R}^T = \begin{pmatrix} \boldsymbol{Q}^T & \mathbf{0} \end{pmatrix}$. Note that Q^T has the same number of rows as the matrix C. Since not any vertex of B will be connected to C, the number of columns of Q^T equals to the number of vertices of B, which have an adjacent vertex in C.

We denote with $\mathcal{T}_{1,A}(C, B, Q, p)$ resp. with $\mathcal{T}_{2,A}(C, B, Q, p)$ the graph described by $\mathcal{T}_{1,A}$ resp. by $\mathcal{T}_{2,A}$, where

$$\mathcal{T}_{1,A} = \begin{pmatrix} \boldsymbol{C} & \mathbf{J}_{1,p} \otimes \boldsymbol{R}^T \\ \mathbf{J}_{p,1} \otimes \boldsymbol{R} & \mathbf{I}_p \otimes \boldsymbol{B} \end{pmatrix}, \mathcal{T}_{2,A} = \begin{pmatrix} \boldsymbol{C} & \mathbf{J}_{1,p} \otimes \boldsymbol{R}^T \\ \mathbf{J}_{p,1} \otimes \boldsymbol{R} & \mathbf{I}_p \otimes \boldsymbol{B} + (\mathbf{J}_{p,p} - \mathbf{I}_p) \otimes \mathbf{I} \end{pmatrix}.$$

In the following we define a sequence of graphs, where each graph is of the form $\mathcal{T}_{1,A}(C, B, Q, p)$ or $\mathcal{T}_{2,A}(C, B, Q, p)$. Note, that R describes Q in a unique way, so $\mathcal{T}_{1,A}$ is well-defined.

Definition 1. *Let $(G_k)_{1 \le k < \infty}$ be a sequence of graphs defined as follows. There exists a sequence of matrices $(C_k)_{1 \le k < \infty}, (Q_k)_{1 \le k < \infty}$ resp. of integers $(p_k)_{1 \le k < \infty}$ for any $1 \le k < \infty$, where $G_k = \mathcal{T}_{1,A}(C_k, Q_k, G_{k-1}, p_k)$. Q_k represents the edges between the core C_k of G_k and the core C_{k-1} of G_{k-1}; Q_k has the same number of rows as C_k and the same number of columns as C_{k-1}. There are no edges between C_k and $G_{k-1} \setminus C_{k-1}$.*

The sequence $(G'_k)_{1 \le k < \infty}$ is defined in a similar way as $(G_k)_{1 \le k < \infty}$. For any $1 \le k < \infty$ there exists a sequence of matrices $(C'_k)_{1 \le k < \infty}$, $(Q'_k)_{1 \le k < \infty}$ and a sequence of integers $(p_k)_{1 \le k < \infty}$ so that $G'_k = \mathcal{T}_{2,A}(C'_k, Q'_k, G'_{k-1}, p_k)$ where Q'_k represents the edges between the core C'_k of G'_k and the core C'_{k-1} of G'_{k-1}. There is no edge between C'_k and $G'_{k-1} \setminus C'_n$.

To show how to calculate the eigenvalues of the graphs G_n resp. G'_n defined above we have to define the graph class $\mathcal{M}_A(C_1, Q_{1,2}, C_2, Q_{2,3} \ldots, C_n)$.

Definition 2. *Let $C_1, C_2, \ldots C_k$ be a sequence of matrices with $1 \leq k < \infty$. Let $Q_{1,2}, Q_{2,3}, \ldots, Q_{k-1,k}$ be also a sequence of matrices.*

$\mathcal{M}_A(C_1, Q_{1,2}, C_2, \ldots, Q_{k-1,n}, C_k)$ denotes the graph with a block tri-diagonal adjacency matrix of the form

$$\begin{pmatrix} C_k & Q_{k-1,k}^T & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ Q_{k-1,k} & C_{k-1} & Q_{k-2,k-1}^T & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & Q_{k-2,k-1} & C_{k-2} & \cdots & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & C_2 & Q_{1,2}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & Q_{1,2} & C_1 \end{pmatrix}.$$

In the following lemma we show that the eigenvalues of G_k defined in definition 1 can be reduced to the eigenvalues of some graphs $\mathcal{M}_A$ defined in definition 2.

Lemma 3. *The spectrum of $G_k = \mathcal{T}_{1,A}(C_k, Q_k, G_{k-1}, p_k)$ is the union of the spectra of n graphs $M_1, \ldots, M_k$, where $M_i = \mathcal{M}_A(C_i, \sqrt{p_i} \cdot Q_i, C_{i-1}, \sqrt{p_{i-1}} \cdot Q_{i-1}, \ldots, C_1)$ for any $1 \leq i \leq k$.*

Furthermore the spectrum of $G'_k = \mathcal{T}_{2,A}(C'_k, Q'_k, G'_{k-1}, p)$ is the union of the spectra of some graphs $M_{i,j}$, $1 \leq i \leq k$ and $1 \leq j \leq k - i$ for $i \neq k$ and $j = 1$ for $i = k$. The graph $M_{i,j} = \mathcal{M}_A(C_{i,j,0}, \sqrt{p} \cdot Q'_i, C_{i,j,1}, \ldots, C_{i,j,i-1})$, where $C_{i,j,l} = C'_{i-l} + (i - k + p(j-1) + l(p-1)) \cdot I$.

Proof. The first statement of the lemma can be proved by induction. We can apply lemma 2 on G_k and obtain, that its eigenvalues are the union of the spectra of G_{k-1} and of the graph $\mathcal{T}_{1,A}(C'_{k,k-1}, Q_{k-1}, G_{k-2}, p_{k-1})$, where $C'_{k,k-1} = \mathcal{M}_A(C_k, \sqrt{p_k} \cdot Q_k, C_{k-1})$. As an example we present the matrix after applying 2 times lemma 2 on G_k for $p_{k-2} = 2$ and obtain

$$\begin{pmatrix} C_k & \sqrt{p_k} Q_k^T & 0 & 0 & 0 \\ \sqrt{p_k} Q_k & C_{k-1} & \sqrt{p_{k-1}} Q_{k-1}^T & 0 & 0 \\ 0 & \sqrt{p_{k-1}} Q_{k-1} & C_{k-2} & R_{k-2}^T & R_{k-2}^T \\ 0 & 0 & R_{k-2} & G_{k-3} & 0 \\ 0 & 0 & R_{k-2} & 0 & G_{k-3} \end{pmatrix}.$$

Now we assume that if we apply i times lemma 2, the eigenvalues are the union of the spectra of $G_{k-1} \ldots G_{k-i}$ and of the graph

$$\mathcal{T}_{1,A}(C^*_{k,k-1,\ldots,k-i}, Q^*_{k-i}, G_{k-i-1}, p_{k-i}),$$

with

$$C^*_{k,k-1,\ldots,k-i} = \mathcal{M}_A(C_k, \sqrt{p_k} \cdot Q_k, C_{k-1}, \ldots, \sqrt{p_{k-1}} \cdot Q_{k-1} C_{k-i})$$

and $Q^{*T}_{k-i} = \begin{pmatrix} 0 \\ Q^T_{k-i} \end{pmatrix}$. After applying lemma 2 once again we obtain, that the spectrum of G_k can be obtained from the spectra of $G_{k-1} \ldots G_{k-i-1}$ and of the spectrum of the graph

$$\mathcal{T}_{1,A}(C^*_{k,k-1,\ldots,k-i-1}, Q^*_{k-i-1}, G_{k-i-2}, p_{k-i-1}),$$

where

$$C^*_{k,k-1,\ldots,k-i-1} = \mathcal{M}_A(C_k, \sqrt{p_k} \cdot Q_k, C_{k-1}, \ldots, C_{k-i-1})$$

and $Q^{*T}_{k-i-1} = \begin{pmatrix} 0 \\ Q^T_{k-i-1} \end{pmatrix}$. After k such steps we obtain the first statement of the lemma. The second statement can be proved in a similar way. □

A similar lemma can be also formulated for the Laplacian eigenvalues of such graphs.

3 Spectra of scalable and non-scalable topologies

In this section we deal with some known families of graphs used as interconnection networks. We begin with the Butterfly network, which is designed to have many favorable properties for distributed computing, such as small maximal degree and diameter, large connectivity etc. Next, we turn our attention to scalable families of graphs. First we construct the graphs $T_{(d,n)}$ with $O\left(\left(\frac{\log n}{\log d}\right)^2\right)$ distinct eigenvalues of both adjacency and Laplacian matrix. Since trees are not well suited as interconnection topologies we define a new graph class with much better network properties. To compute the eigenvalues, we use the technique presented in the previous section.

3.1 The spectrum of the Butterfly graph

The spectral properties of some interconnection networks like rings, tori, hypercubes [7] or DeBruijn graphs [9] have been investigated. Now we present the adjacency and Laplacian spectrum of the Butterfly without wrap-around edges.

The Butterfly graph $BF_{(k)}$ consists of $k+1$ columns, each column containing 2^k vertices labeled with unique binary strings of length k. An edge connects two vertices in $BF_{(k)}$ if and only if they are in consecutive i-th and $(i+1)$-st columns and their labels are either equal or differ only in the i-th bit.

Theorem 1. *The adjacency spectrum of the Butterfly graph $BF_{(k)}$ is*

$$Sp_A(BF_{(k)}) = \left\{ 4\cos\left(\frac{\pi i}{j+1}\right) \mid 1 \le i \le j \le k+1 \right\}$$

Proof. The Butterfly graph $BF_{(k)}$ can be recursively constructed by taking two copies of $BF_{(k-1)}$, adding one column of 2^k vertices and connecting these vertices to both $BF_{(k-1)}$'s. Thus the adjacency matrix of the Butterfly graph $BF_{(k)}$ has the structure from Lemma 3 of $\mathcal{T}_{1,A}(C_k, BF_{k-1}, Q_k, 2)$ with $\boldsymbol{C}_k = \mathbf{0}$ and $\boldsymbol{Q}^T_k = \begin{pmatrix} \mathbf{I}_{2^{k-1}} \\ \mathbf{I}_{2^{k-1}} \end{pmatrix}$. Using Lemma 3, we get $Sp_A(BF_{(k)}) = \cup_{i=0}^{k} \{Sp(\boldsymbol{M}_i)\}$. The matrix $\boldsymbol{M}_i$ can be viewed as an adjacency matrix of a weighted binary tree with all edges of weight $\sqrt{2}$. After permuting the vertices, Lemma 3 can be used again with $\boldsymbol{C}_k = (0)$, $p_k = 2$ and $\boldsymbol{Q}_k = (\sqrt{2})$. Thus $Sp_A(BF_{(k)}) = \cup_{i=0}^{k} \{Sp(\boldsymbol{M}_i)\} = \cup_{i=0}^{k} \cup_{j=0}^{i} \{Sp(\boldsymbol{M}'_j)\}$. The

matrix $\boldsymbol{M}'_j$ can be viewed as an adjacency matrix of a weighted path with $j+1$ vertices where all edges have weight 2, thus using [7] $Sp(\boldsymbol{M}'_j) = 2Sp(P_{j+1}) = 4\left\{\cos\left(\frac{\pi l}{j+2}\right) \mid l = 1, ..., j+1\right\}$. □

Using similar techniques the Laplacian spectrum of the Butterfly can ce also computed.

Theorem 2. *The Laplacian spectrum of the Butterfly graph $BF_{(k)}$ is*

$$Sp_{\Delta}(BF_{(k)}) = \left\{4 - 4\cos\left(\frac{\pi i}{j+1}\right) \mid 0 \le i \le j \le k\right\} \cup \\ \cup \left\{4 - 4\cos\left(\frac{\pi(2i-1)}{2j+1}\right) \mid 1 \le i \le j \le k\right\}$$

3.2 Adjacency and Laplacian spectrum of scalable families of graphs

Now we turn our attention to scalable families of graphs. In the sequel, n will always denote the number of vertices. Let us define the graph class $T_{(d,n)}$ as follows.

Definition 3. *The tree $T_{(d,n)}$ is a rooted tree defined recursively. $T_{(d,1)}$ contains only the root vertex. For $n < d+1$, $T_{(d,n)}$ is a star with one root and $n-1$ leaves connected to it. For $n > d$, $T_{(d,n)} = (V, E)$ is constructed as follows. Let $s = \lfloor \frac{n-1}{d} \rfloor$ and $q = n - 1 - ds$. Let $(V_1, E_1), ..., (V_d, E_d)$ be d disjoint copies of $T_{(d,s)}$ with respective roots r_1,...,r_d. Then $V = \{r\} \cup \{v_1, ..., v_q\} \cup \bigcup_{i=1}^{d} V_i$ and $E = \{(r, v_i) \mid i = 1...q\} \cup \{(r, r_i) \mid i = 1..d\} \cup \bigcup_{i=1}^{d} E_i$.*

Informally, constructing a tree $T_{(d,n)}$ involves setting one vertex r as a root, then dividing the remaining $n-1$ vertices evenly and constructing a number of copies of $T_{(d,s)}$. The roots of these copies are connected to r. Remaining vertices are added as vertices with degree 1 connected to r.

Remark 1. The graph $T_{(d,n)}$ has n vertices and the maximal degree at most $2d+1$.

Theorem 3. *The graph $T_{(d,n)}$ has at most $O\left(\left(\frac{\log n}{\log d}\right)^2\right)$ different eigenvalues.*

The proof of this theorem follows from Lemma 3 and it will be omitted because of space limitations.

The choice of the parameter d results in graphs with different properties. Depending on the application, one may ask for either minimal number of eigenvalues or for the minimal value of $deg \cdot |Sp|$, where deg is the maximal degree of the graph. For getting a small number of eigenvalues, some reasonable large value of d should be chosen (e.g., $d = O(\log n)$), and for setting a small value of the product, some small value should be chosen (e.g., $d = 2$).

However, trees are extremely ill-suited for the application of load balancing because of their poor connectivity properties. To overcome this weakpoint, we give another construction of a graph $H_{(n)}$ with $O\left((\log n)^3\right)$ distinct eigenvalues which is much better suited as a topology for load balancing.

Definition 4. *Every graph $H_{(n)}$ has a set of distinguished vertices (core) denoted by $C(H_{(n)})$. The graph $H_{(n)}$ is defined recursively as follows. $H_{(1)} = C(H_{(1)}) = K_1$, $H_{(2)} = C(H_{(2)}) = K_2$. For $H_{(n)} = (V, E)$, $n > 2$, let $H_{(n)}$ be of the form $\mathcal{T}_{2,A}(C_n, H_{(\lfloor \frac{n-1}{2} \rfloor)}, Q_n, 2)$, where*

$$C_n = \begin{cases} K_2 \text{ if n is even} \\ K_1 \text{ if n is odd} \end{cases},$$

and $Q_n = J$.

Informally, to construct $H_{(n)}$, first construct two copies of $H_{(\lfloor \frac{n-1}{2} \rfloor)}$ and connect the corresponding vertices. The remaining 1 or 2 vertices of $H_{(n)}$ form its core and are connected mutually and to all vertices from the cores of both $H_{(\lfloor \frac{n-1}{2} \rfloor)}$s (see Figure 3.2). In consequence, $H_{(n)}$ has a vertex degree of at most $\log n + 5$.

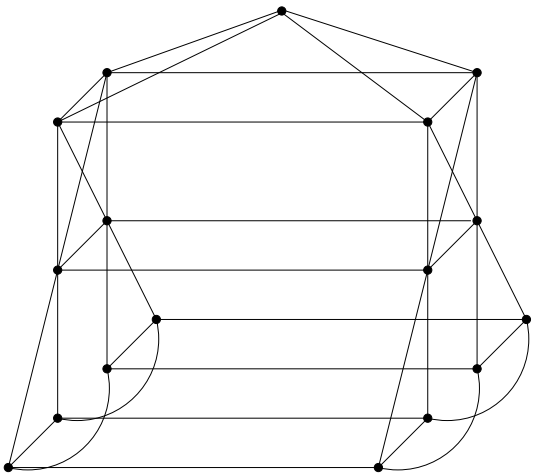

Fig. 1. The graph $H_{(17)}$.

Theorem 4. *The graph $H_{(n)}$ has at most $O(\log^3 n)$ distinct eigenvalues.*

This theorem can be proved using Lemma 3 and it will be omitted because of space limitations.

4 Adjacency spectrum

In this section we improve the previous results for the case of the adjacency spectrum. Using another technique we construct a scalable family of graphs $G_{(n)}$ with $O\left((\log n)^2\right)$ distinct eigenvalues.

The graph $G_{(n)}$ is constructed by taking a hypercube, subdividing each edge and then replicating the vertices from the original hypercube a number of times (see Figure 4).

Definition 5. *Let d $(d > 0)$ satisfy the inequality $2^{d-1}(d+2) \leq n < 2^d(d+3)$. Let $\{\Delta_i\}_{i=0}^{d}$ be a sequence defined as follows:*

$$\Delta_i = \begin{cases} 0 & \text{for } i > \left\lfloor \frac{d}{2} \right\rfloor \\ \left(n - 2^{d-1}(d+2)\right) \mod 2^d & \text{for } i = \left\lfloor \frac{d}{2} \right\rfloor \\ \Delta_{i+1} \bmod \binom{d}{i+1} & \text{for } 0 \leq i < \left\lfloor \frac{d}{2} \right\rfloor \end{cases}$$

where $a \bmod b = a - b \cdot \left\lfloor \frac{a}{b} \right\rfloor$.

Let $Q(d)$ be a hypercube. We shall refer to the vertices as binary strings from $\{0,1\}^d$. The k-th level is defined as $\mathcal{L}_k = \left\{x \in \{0,1\}^d \mid \#_1(x) = k\right\}$.

The graph $G_{(n)}$ is defined as follows. Consider the graph $S(Q(d))$ with $2^{d-1}(d+2)$ vertices obtained from the hypercube $Q(d)$ by subdivision of each edge. For each node x from the original graph $Q(d)$ add a set $\mathcal{V}_x$ of isolated vertices of cardinality $|\mathcal{V}_x| = \left\lfloor \frac{\Delta_k}{\binom{d}{k}} \right\rfloor + \left\lfloor \frac{n - 2^{d-1}(d+2)}{2^d} \right\rfloor$ where $x \in \mathcal{L}_k$. For each $y \in \mathcal{V}_x$ add edges (y, v) for all edges (x, v) from $S(Q(d))$.

Now, the diameter of a hypercube $Q(n)$ is $\log(n)$ and from the construction of the graph follows:

Remark 2. The graph $G_{(n)}$ $(n > 2)$ has n vertices and $diam(G) \leq 2 \log n$.

The maximal degree of $G_{(n)}$ follows also from its definition.

Remark 3. The maximal degree of $G_{(n)}$ $(n > 2)$ is at most $3 \log n + o(\log n)$.

To compute the number of distinct eigenvalues of $G_{(n)}$ we need the further two lemmas.

Lemma 4. *[7] Let $\boldsymbol{M}$ be a non-singular square matrix, then*

$$\begin{vmatrix} \boldsymbol{M} & \boldsymbol{N} \\ \boldsymbol{P} & \boldsymbol{Q} \end{vmatrix} = |\boldsymbol{M}| \cdot |\boldsymbol{Q} - \boldsymbol{P}\boldsymbol{M}^{-1}\boldsymbol{N}|$$

Lemma 5. *Let G be a graph obtained from the hypercube $Q(d)$ as follows. For every vertex x add a self-loop with weight w_k^2, where $x \in \mathcal{L}_k$. Each edge connecting vertices on levels k, $k+1$ has weight $w_k \cdot w_{k+1}$. Then the graph G has at most $O(d^2)$ distinct eigenvalues.*

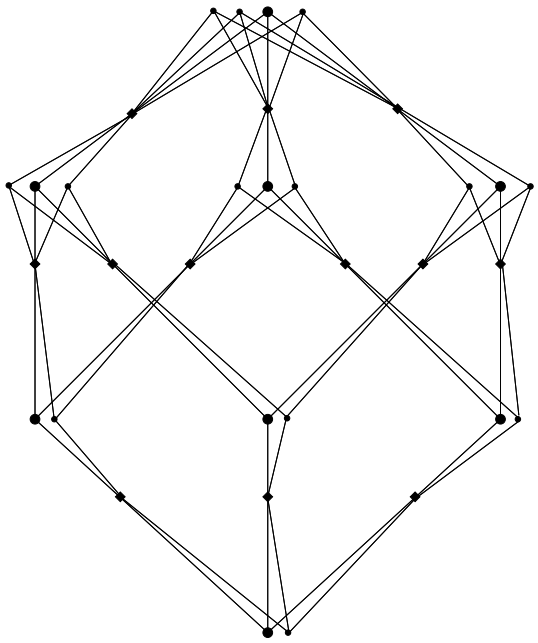

Fig. 2. The graph $G_{(33)}$.

The proof of this lemma is omitted because of space limitations.

Theorem 5. *The adjacency matrix of* $G_{(n)}$ $(n > 2)$ *has* $O(\log^2 n)$ *different eigenvalues* .

Proof. Let $\boldsymbol{A}$ be the adjacency matrix of $G_{(n)}$. Consider an arbitrary vertex x from the original hypercube $Q(d)$ together with all vertices from $\mathcal{V}_x$. Let $\boldsymbol{C}$ be the adjacency matrix of a graph obtained from $G_{(n)}$ by removing vertices $\{x\} \cup \mathcal{V}_x$. Clearly $\boldsymbol{A}$ is of the form of Lemma 2. After applying Lemma 2 iteratively to the sets $\{x\} \cup \mathcal{V}_x$ for each $x \in Q(d)$, we get a matrix $\boldsymbol{A}'$ with the same spectrum as $\boldsymbol{A}$ (except possibly the eigenvalue 0). The matrix $\boldsymbol{A}'$ can be viewed as the adjacency matrix of a weighted graph G' defined as follows. Consider a subdivided hypercube $S(Q(d))$. Each edge incident with a vertex x from $Q(d)$ has a weight $\sqrt{1 + |\mathcal{V}_x|}$.

Clearly, $\boldsymbol{A}'$ is of the form $\boldsymbol{A}' = \begin{pmatrix} \mathbf{0} & \boldsymbol{R}^{\mathsf{T}}\boldsymbol{D} \\ D\boldsymbol{R} & \mathbf{0} \end{pmatrix}$, where $\boldsymbol{R} \in \mathbb{R}^{2^d \times d2^{d-1}}$ is the vertex-edge incidency matrix of $Q(d)$ and $\boldsymbol{D} \in \mathbb{R}^{2^d \times 2^d}$ is a diagonal matrix $\boldsymbol{D} = (d_i\delta_{ij})_{i,j} = \left(\sqrt{1 + |\mathcal{V}_i|}\delta_{ij}\right)_{i,j}$. Now using Lemma 4 we get $|\lambda\mathbf{I} - \boldsymbol{A}'| = \begin{vmatrix} \lambda\mathbf{I} & -\boldsymbol{R}^{\mathsf{T}}\boldsymbol{D} \\ -\boldsymbol{D}\boldsymbol{R} & \lambda\mathbf{I} \end{vmatrix} = |\lambda\mathbf{I}| \cdot |\lambda\mathbf{I} - \boldsymbol{D}\boldsymbol{R}\frac{1}{\lambda}\boldsymbol{R}^{\mathsf{T}}\boldsymbol{D}| = \lambda^{d2^{d-1}} \cdot |\frac{1}{\lambda}(\lambda^2\mathbf{I} - \boldsymbol{D}\boldsymbol{R}\boldsymbol{R}^{\mathsf{T}}\boldsymbol{D})| = \lambda^{d2^{d-1}-2^d} \cdot |\lambda^2\mathbf{I} - \boldsymbol{D}\boldsymbol{R}\boldsymbol{R}^{\mathsf{T}}\boldsymbol{D}|$. Using the fact that $\boldsymbol{R}\boldsymbol{R}^{\mathsf{T}} = \boldsymbol{Q} + d\mathbf{I}$, where $\boldsymbol{Q}$ is the

adjacency matrix of hypercube $Q(d)$, we can conclude that $G_{(n)}$ has at most $2m+1$ distinct eigenvalues, where m is the number of distinct eigenvalues of a graph G'' defined as follows. Consider a hypercube $Q(d)$. For each $k = 0, ..., d$, let $w_k = \sqrt{1 + |\mathcal{V}_k|}$. Add to each vertex x from $Q(d)$ a self-loop with weight w_k^2, where $x \in \mathcal{L}_k$. Each edge connecting vertices in levels k and $k+1$ has weight $w_k \cdot w_{k+1}$. Now using Lemma 5 we can show, that G'' has only $O(\log^2 n)$ distinct eigenvalues.

5 Acknowledgment

The authors would like to thank Jean-Pierre Tillich and David Wales for the fruitful discussions and their assistance w.r.t. the eigenvalues of the Star graph.

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