Possible Worlds and Resources: The Semantics of BI*

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Abstract

In a previous paper, O'Hearn and Pym introduced a new substructural logic, the logic **BI** of bunched implications. Herein, motivated by modelling *resources*, we investigate the possible worlds semantics of propositional **BI**. The basic idea is to regard a world as a resource, or a collection of resources, and so give a semantics to **BI**'s additive and multiplicative connectives. The semantics can be formulated at three levels. At the level of *preordered monoids*, we obtain (soundness and) completeness in the absence of inconsistency ($\perp$, the unit of $\vee$). Moving to *topological monoids* we obtain completeness including inconsistency. We give a detailed proof of soundness and completeness at the level of *Grothendieck sheaves* on a preordered monoid. We give computational examples of models, in which the model of resources is explicit, based on Petri nets, process calculus terms and a model of the computer store.

1 Introduction

In a previous paper, O'Hearn and Pym introduced a new substructural logic, **BI**, the logic of bunched implications [1]. There, the perspective was primarily proof-theoretic: we showed how bunches, which are tree-like contexts, could be used to freely mix together a substructural implication $-\!\!*$ and a conventional additive (intuitionistic or classical) implication $\rightarrow$. **BI** includes all of the connectives ($\top$, $\wedge$, $\perp$, $\vee$, $\rightarrow$) of intuitionistic logic, alongside those (I, $*$, $-\!\!*$) of a basic substructural logic. Here, $*$ is a commutative conjunction that lacks Weakening and Contraction, and $-\!\!*$ is its implication adjoint. A Boolean variant of **BI** is obtained by adding the law of reductio ad absurdum to the intuitionistic system.

The purpose of this paper is to explore in some detail the possible worlds semantics of propositional **BI** sketched in [1]. A similar analysis for predicate **BI** [2], building on the present analysis, is treated in [3].

Our first reason for doing this is simple: the possible worlds semantics shows how the logic has many interesting and naturally occurring models. Furthermore, although they can sometimes provide semantics for proofs, many of these models make much more sense from a *truth-based perspective*, in which one views a formula as making a declarative statement about some state of affairs.

Our second reason is comparatively contentious: we will argue that the semantics shows how **BI**'s formulæ can be understood in terms of *resources*; that it may be regarded, in short, as a *resource semantics*.

One of the definitions offered in the Oxford English Dictionary says just that resource is "the means available to achieve an end". In this sense, the concept is very broad and cuts across many disciplines: One has only to think of bank accounts, petroleum or machines in a factory.

Resource has a basic rôle in computational systems, where it is a central organizing concept that guides development. In his seminal text on operating systems (which includes an unusually lucid discussion of resource) [4], Brinch Hansen states: "The word *resource* covers physical components, processes, procedures and data structures; in short, any object referenced by computations." Particularly important here is the use of "references", which leads on to

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the pivotal notion of *resource sharing* that occurs when different computations reference the same resource [5, 4, 6]. Examples of shared resources include storage, files, external devices and processor time.

Our approach in this paper will be to take resource as a primitive in our proposed semantics. The basic idea is to use a possible worlds formalism, where the worlds are thought of as resources, or collections of resources. In brief, we think of the forcing relation $m \models \varphi$ as asserting that the resources m "make φ true". This means not that φ is true on its own but rather that it is true when we confine reference to resources to those in m.

To make this concrete, suppose that "resource" means "area of computer memory". Then we can describe an independent conjunction, that works by decomposing memory into disjoint portions. That is, $\varphi * \psi$ is true just when the current resource (area of memory) can be decomposed into constituents in such a way as to make φ true of one constituent and ψ true of the other. It should be clear that $\varphi * \varphi$ and φ are not equivalent for this reading, for

"this area of memory contains three cells"

is true when the current memory has three cells but

"this area of memory contains three cells" $*$ "this area of memory contains three cells"

is not, because there needs to be three in each of two disjoint portions.

From a proof-theoretic point of view, if we are to deny the equivalence of φ and $\varphi * \varphi$ then we must restrict use of the structural rule of Contraction.

$$\frac{\Gamma, \varphi, \varphi \vdash \psi}{\Gamma, \varphi \vdash \psi} \text{ Contraction}$$

(That is, assuming $*$ is a connective that internalizes the comma.) So we are talking about the conjunction for a substructural logic here.

The intuitive connection between resource and substructural logic is not new. Beginning in the early 70s, relevant logicians have developed a possible world semantics of substructural logics [7], which can be seen as a resource semantics [8, 9]. More recently, Girard has presented a novel explanation of the structural rules [10, 11], which has been dubbed "resource conscious" by others; most notably, the absence of Contraction is interpreted as disallowing the free duplication of pieces of data.

The semantics in this paper might be considered as a revisiting of the possible world semantics of substructural logic, modified by two further factors. One of these factors is a spatial view of possible worlds semantics, which developed in the context of the semantics of storage, where sharing rather than duplication is the major concern [12, 13, 14, 15]. The demands of the specific models there require a logic different from extant linear or relevant logics. A similar view, from a type-theoretic, or logical frameworks, perspective, can be seen in [16, 17].

The other, more foundational, motivation comes from a proposal of resource as a primitive concept, among others, in the emerging discipline of *informatics*, which can be defined as the *science of the structure, complexity and communication of information* [3]. The treatment of resource in our semantics emphasises its structure and rôle in communication, in contrast to the analysis found in complexity theory. We suggest that the significance of our treatment of resource stretches well beyond the strongly computing-related examples discussed herein.

We emphasize that the techniques used in this paper have been available for some time. For the interpretation of substructural connectives, in the 60s Lambek presented models of his "syntactic calculus", based on the observation that the set of the subsets of a monoid has a residuated structure [18]. Then this observation – with commutativitity and idempotency requirements to account for the structural rules of Exchange and Contraction – was used by Routley and Urquhart in the 70s as the basis for what is sometimes referred to as the "operational semantics" of relevant logics. A slightly broader form of this semantics was devised by Routley and Meyer, taking a ternary relation on worlds as primitve rather than a (commutative) monoid. (We refer to [7] for references to the relevant logic literature.) Also in the early 70s, Day studied closed structure in functor categories. His notion of premonoidal category is closely related to the ternary relation semantics, although in a much more abstract setting [19]. Day's treatment of multiplicatives generalizes those given by Lambek and the relevantists. In this paper we have chosen to take the monoid rather than ternary relation semantics as primitive, partly because the latter is difficult to justify in intuitive terms. Mathematically, however, we could choose either.

As for the intuitionistic connectives, we will give two versions of the semantics, one based on Kripke's semantics of intuitionistic logic and another based on sheaves for Grothendieck topologies. The latter provides a very general semantics of intuitionistic logic with many useful models, and it enables a simple proof of completeness based on a Yoneda argument [20]; we adapt that here, by incorporating substructural connectives. To keep this part simple, we work exclusively with preorders rather than categories. We refer the reader to [21] for a pleasantly elementary account of Grothendieck sheaf semantics (as well as other intuitionistic semantics), in the simplified setting of topologies over Heyting algebras. (That the semantics we present obtains from combining standard semantics is part of the point, and provides an independent motivation for the formulation of **BI**.)

In presenting this material we have opted to take a gradual, partly tutorial style, where we build up to our ultimate (Grothendieck topological) semantics rather than go there in one step. We adopt this approach firstly because we feel that it may make the material more accessible, and secondly because we are aware of a common criticism of substructural logics, and their semantics. That is, that there are too many substructural logics, a veritable zoo, and their semantics does not provide a way to filter out those that are more important, or interesting. In acknowledging this criticism, it is well to consider two logics that do stand out: syntactic calculus [22], and linear logic [10]. Both systems have not only a mathematical semantics but also an intuitively compelling informal interpretation of the semantics (based on linguistics for syntactic calculus and a resource reading of proofs for linear logic). And each system has numerous interesting specific models that fit with the informal readings. These points are primarily responsible for the wide interest generated by syntactic calculus and linear logic, well outside circles of substructural logicians.

With this historical perspective in mind, our main purpose is not just to present a completeness result, but is to make the case for the resource interpretation of **BI**, and to back it up with a number of specific models that illustrate the interpretation. The reader who prefers a more streamlined mathematical presentation is advised to skip directly to § 6.3.

The further contents of the paper are as follows. We begin in the next section by presenting an informal "vending machine" example, which sets the frame for what follows. We continue by giving algebraic models and a calculus for propositional **BI**. We then present the first, elementary, version of the possible worlds semantics, together with brief examples based on Petri nets and CCS. These elementary models use Kripke's semantics to treat the intuitionistic fragment (in some cases this reduces to the boolean algebra structure of a powerset).

After presenting this material, we consider the technical issue of completeness. **BI**'s calculus forces a rather delicate treatment of *inconsistency* which forces us to refine the elementary preordered commutative monoid semantics to exploit technically its inherently topological structure.

We move on to give a more substantial example model based on pointers. It is somewhat intricate, so we develop it in more detail than other examples. The model is particularly important, however, because it is an example in which *resource sharing* is explicit (it was not so in the cost example and is less clear in the Petri and CCS models).

Our technical development culminates with a less elementary semantics. It is again based on (preordered) commutative monoids but this time topological concepts (topological monoids and sheaves) are brought to bear in order to give a more refined account of inconsistency and completeness. Our most sophisticated semantics, for which we give a detailed proof of completeness, is based on Grothendieck sheaves on preordered monoids.

It is worth mentioning that two further models have recently been developed by Cardelli, Gordon and Ghelli in their work on Ambient Logic [23, 24], which is an extension of (Boolean) **BI**. We do not present these models here partly for space reasons, and also because there is considerable overlap with the Petri net, CCS and pointer models. These models fit well with the resource interpretation and provide excellent examples of our basic point of the desirability of having structurally full-strength additives, alongside substructural connectives.

2 Example: What Can I Afford?

In this example the resources are coins, which can be used to buy chocolates or candy from a vending machine (with thanks to C.A.R Hoare and J.-Y. Girard for their related examples). A proposition is a statement about cost and the judgement of consequence is read as follows:

$$\varphi \vdash \psi \quad : \quad \text{If I have enough money to make } \varphi \text{ true, then I have enough to make } \psi \text{ true.}$$

We posit meanings for two conjunctions and implications.

$\varphi \to \psi$	:	If I were to obtain enough money to make φ true, then I should also have enough to make ψ true;
$\varphi \wedge \psi$	:	The money I have got is enough to make φ true and enough to make ψ true;
$\varphi \mathrel{-\!\!*} \psi$	:	If you were to give me enough to make φ true then, combined with what I have already got in my pocket, I should have enough to make ψ true; and
$\varphi * \psi$	:	I can use part of my money to make φ true and have enough left over to make ψ true (and vice versa).

We hope the reader can take these informal descriptions in good spirit. In any case, we will not here pursue a philosophical analysis of what it is for a resource to make a proposition true and instead push on with the example.

Given these readings the following judgements say that for one coin I can buy a candy and for two I can buy a chocolate.

(A1) $coin \vdash candy$, and

(A2) $coin * coin \vdash choc$,

where the basic propositions are

- *coin* : I have (at least) one coin in my pocket,
- *choc* : I have enough to buy a chocolate, and
- *candy* : I have enough to buy a candy.

(Here we are regarding (A1) and (A2) as axioms, so $coin \nvdash choc$: we indend that you must have *at least* two coins to buy a choclate.)

With this as background, we now move on to consider some judgements which illustrate the most important, or unusual, consequences of the readings. Certainly, the most distinctive feature of **BI** is its joint treatment of the two implications. As an example of how $\mathrel{-\!\!*}$ works, we certainly expect

- $coin \vdash coin \mathrel{-\!\!*} choc$

because if I have a coin in my pocket, and if you give me another, then I will have enough to buy a chocolate. However,

- $coin \nvdash coin \to choc$

because a single coin is not enough to buy a chocolate.

The reader will detect some similarity here with Girard's well-known "Marlboro's and Camels" reading of linear logic [11]. However, the divergences are both more interesting than the similarities and illustrate how great is the difference between **BI** and linear logic. First, and foremost, Girard's reading is about "proofs-as-actions" where, for example,

- *choc*: the (type of the) act of buying a chocolate.

In contrast, our reading is not about proofs. We do not regard a proposition as a resource and (so) a proof as a way to manipulate resources. Rather, the reading is completely declarative: a proposition is a statement about the world whose judgement of truth may involve consideration of resources.

Second, the difference is not merely one of emphasis but can be seen on the level of logical consequence. For instance,

- $coin \mathrel{-\!\!*} choc \nvdash coin \to choc$

is something we would expect, because $coin \mathrel{-\!\!*} choc$ is true when you have one coin in your pocket but $coin \to choc$ is not. In linear logic, however, where $\varphi \to \psi$ is rendered as $!\varphi \multimap \psi$, one gets

- $coin \multimap choc \vdash coin \to choc \quad (= !coin \multimap choc)$

no matter what *coin* and *choc* are, because one can compose on the left with dereliction $!\varphi \vdash \varphi$.

There are other examples in **BI** which violate the "use once" idea from linear logic (here I is the unit of the multiplicative conjunction, $*$):

- $I \vdash (coin \wedge (coin \to choc)) \mathrel{-\!\!*} choc$, and
- $I \vdash coin \mathrel{-\!\!*} ((coin \to coin \to choc) \to choc)$.

Now these judgements seem wrong from the point of view of linear logic because

- $I \not\vdash (coin\,\&(!coin \multimap choc)) \multimap choc$, and
- $I \not\vdash coin \multimap\, !(!coin \multimap (!coin \multimap choc)) \multimap choc$.

The first case would violate the idea that a linear function of type $A\&B \multimap C$ must use one of its input components but not both, and the second would violate the idea that a linear function cannot use its argument twice. However, if one discards this perspective and thinks declaratively, using the reading of formulæ advanced in this section, then the truth of **BI**'s judgements is straightforward.

In **BI**, the proof of the last judgement, when viewed as an $\alpha\lambda$-term [1, 15, 3], does indeed use its argument twice. Indeed, in [1] we advanced a resource reading of proofs to justify this judgement; the declarative justification is much more immediate.

All of the true judgements we have claimed in this section, and non-judgements, will hold in the semantics of this paper. (That is, when we assume the (A1) and (A2) as axioms.) All told, what this indicates is that **BI** and linear logic are *incomparable* extensions of intuitionistic logic, and the basic substructural logic (sometimes called BCI logic or multiplicative intuitionistic linear logic). That is, when we consider formulæ which mix additives and multiplicatives, we have some judgements that hold in **BI** but not linear logic; and some the other way around.

We agree that the "proofs-as-actions" reading of linear logic is very appealing and that it gives a consistent way of understanding the semantics of the judgements above given by linear logic's consequence relation. Equally, we claim that the declarative resource reading gives a clear justification for the exact opposite position on the corresponding judgements, the position taken by **BI**.

From this example we can see a number of assumptions, which we take as primitive.

- An underlying *set* of resources;
- A representative for *zero* resources;
- A way of *combining* resources;
- A way of *comparing* resources.

Mathematically, we will start by considering a preordered commutative monoid, $\mathcal{M} = (M, \cdot, e, \sqsubseteq)$, as the basis for a possible worlds semantics. In the cost example, the monoid is the natural numbers with addition and the usual less-than ordering (reversed); a forcing clause $m \models \varphi$ is regarded as saying that having m coins in your pocket is enough to satisfy φ.

At this point, the reader might ask: is that all ? It is tempting to try to add data to the monoid, corresponding to ideas such as sharing or ownership, or to add axioms to rule out examples that are not "resource-like". We prefer a more conservative approach: Our mode of working is to start small with a simple and well-motivated model, and see how far we can get before assessing limitations. Resource is a multi-faceted notion and it is doubtful that a comprehensive or final logical theory can be found straightaway. A more constrained approach is called for, as it is often easier to refine a simple, well-understood theory than to start with a complex one.

3 BI-algebras and Elementary Models

3.1 An Algebraic Semantics and a Calculus

In this section, we set up the framework for the remainder of the paper. Since we are primarily concerned with truth and provability, rather than the structure of proofs, we present a simple-minded algebraic semantics and associated calculus for **BI**. This presentation of **BI** does not make use of bunches, *i.e.*, **BI**'s tree-structured contexts, the details of which can be found in [1, 3].

In order to motivate the models, however, it is useful to briefly sketch the categorical interpretation which lies at the core of **BI**. Suppose that we are to have a logic with two implications. Then categorically, a natural notion of consequence arises from doubly closed categories, which are categories that possess two closed structures or function spaces. That is, we have a single category with two adjunctions

$$[A * B, C] \cong [A, B \mathrel{-\!\!*} C] \qquad \text{and} \qquad [A \wedge B, C] \cong [A, B \to C]$$

which determine the behaviour of $\mathrel{-\!\!*}$ and $\to$. The algebra models are collapsed versions of these structures, where the additive implication $\to$ corresponds to intuitionistic logic and the multiplicative $\mathrel{-\!\!*}$ to a basic substructural logic.

To describe the models, first recall that Heyting algebras are the algebraic models of intuitionistic propositional logic. A Heyting algebra is a lattice with greatest and least elements in which the meet $a \wedge b$ is *residuated*, which is to say that there is an implication operator, $\to$, satisfying

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c.$$

Second, an algebraic model of a basic substructural logic containing conjunction $*$, unit I and implication $\mathrel{-\!\!*}$ is similar, except that $*$ is not required to be meet, and I is not required to be top. That is, we would require a partial order with a (monotone) commutative monoid structure that is residuated, so that

$$a * b \leq c \quad \text{iff} \quad a \leq b \mathrel{-\!\!*} c.$$

Because we have all of the connectives of intuitionistic logic and the basic substructural logic at the same time, we simply ask for a single algebra that has both kinds of structure.

A **BI**-*algebra is a Heyting algebra equipped with an additional residuated commutative monoid structure.*

Note that the same underlying order is used to describe the residuated structure in both cases.

From this notion of **BI**-algebra, it is straightforward to derive a collection of axioms and rules for proving judgements $\varphi \vdash \psi$, where the formulæ φ and ψ are built from propositional variables, the additive connectives $\to$, $\wedge$, $\top$, $\vee$ and $\bot$, and the multiplicative connectives I, $*$ and $\mathrel{-\!\!*}$. The axioms and rules of this system are those for a presentation of intuitionistic propositional logic, together with the rules for the substructural fragment.

$$\varphi \vdash \varphi$$

$$\varphi \vdash \top \qquad\qquad \bot \vdash \varphi$$

$$\frac{\eta \vdash \varphi \quad \eta \vdash \psi}{\eta \vdash \varphi \wedge \psi} \qquad\qquad \frac{\varphi \vdash \psi_1 \wedge \psi_2}{\varphi \vdash \psi_i}$$

$$\frac{\eta \vdash \psi \quad \varphi \vdash \psi}{\eta \vee \varphi \vdash \psi} \qquad\qquad \frac{\varphi \vdash \psi_i}{\varphi \vdash \psi_1 \vee \psi_2}$$

$$\frac{\chi \wedge \varphi \vdash \psi}{\chi \vdash \varphi \to \psi} \qquad\qquad \frac{\chi \vdash \varphi \to \psi \quad \eta \vdash \varphi}{\chi \wedge \eta \vdash \psi}$$

$$\varphi * (\psi * \chi) \dashv\vdash (\varphi * \psi) * \chi \qquad\qquad \varphi * I \dashv\vdash \varphi \dashv\vdash I * \varphi$$

$$\frac{\chi \vdash \varphi \quad \eta \vdash \psi}{\chi * \eta \vdash \varphi * \psi} \qquad\qquad \varphi * \psi \vdash \psi * \varphi$$

$$\frac{\chi * \varphi \vdash \psi}{\chi \vdash \varphi \mathrel{-\!\!*} \psi} \qquad\qquad \frac{\chi \vdash \varphi \mathrel{-\!\!*} \psi \quad \eta \vdash \varphi}{\chi * \eta \vdash \psi}$$

The reader will recognize here laws for coproducts, products, tensor products, and implicational adjunctions. We say simply that "$\psi \vdash \varphi$ is derivable" to indicate that $\psi \vdash \varphi$ can be proven using this system or, equivalently, when

$[\![\psi]\!] \leq [\![\varphi]\!]$ for all interpretations $[\![-]\!]$ in **BI**-algebras (**BI**-algebras obviously give complete models of the proof system just given.)

This way of formulating the system is unsophisticated proof theoretically, but it is adequate for capturing provability. **BI** can also be presented as a system of natural deduction [1, 2, 3] or, indeed, as a sequent calculus [2, 3], where the two conjunctions are mimicked by two context-combining operations. Normalization and cut elimination results for these calculi are established in [3].

We will give several models in which the additives are treated *classically*. So we define "Boolean **BI**" to be the consequence relation generated by the above rules, plus *reductio ad absurdum*:

$$\frac{\psi \vdash (\varphi \to \bot) \to \bot}{\psi \vdash \varphi}\ RAA.$$

An algebra model for this system is a *Boolean* **BI**-*algebra*, a **BI**-algebra in which the Heyting component is in fact Boolean.

The notion of **BI**-algebra is useful as a reference point but the definition itself does not suggest a declarative way of reading formulæ; neither does it tell us if there are any interesting **BI**-algebras. Possible worlds models address both of these points.

3.2 A Forcing Semantics

The semantics is stated in terms of a judgement form $m \models \varphi$ which says that formula φ is true at, or with respect to, a world m. We start with a preordered commutative monoid, $\mathcal{M} = (M, \cdot, e, \sqsubseteq)$. Here the multiplication $\cdot$ is monotone and equality in the monoid (which the monoid laws hold up to) is that which is given by the equivalence relation $\sqsubseteq \cap \sqsupseteq$.

Given such a monoid, semantic clauses can be given for a form of truth, conjunction and implication as follows:

$$\begin{array}{rll} m \models I & \text{iff} & m \sqsubseteq e \\ m \models \varphi * \psi & \text{iff} & \exists n, n' \in M\,(m \sqsubseteq n \cdot n' \ \text{ and } \ n \models \varphi \ \text{ and } \ n' \models \psi) \\ m \models \varphi \mathrel{-\!\!*} \psi & \text{iff} & \forall n \in M\,(n \models \varphi \ \text{ implies } \ n \cdot m \models \psi) \end{array}$$

Now, the conjunction thus obtained does not admit Weakening or Contraction generally, in that the implications

- if $m \models \varphi * \psi$, then $m \models \varphi$, and
- if $m \models \varphi$, then $m \models \varphi * \varphi$

do not necessarily hold. However, it does have the implicational adjunction

- $m \models (\varphi * \psi) \mathrel{-\!\!*} \chi$ iff $m \models \varphi \mathrel{-\!\!*} (\psi \mathrel{-\!\!*} \chi)$.

Variations on this semantics have been taken as the basis for a number of notions of model for substructural logics (*e.g.*, [25, 10]).

Of course, a substructural logic with only these three connectives is very weak and the way in which other connectives are added is one place where significant divergence occurs. However, a simple point is central: there is already enough structure to interpret all of the connectives of intuitionistic logic, in the style of possible worlds semantics, without adding anything to the basic set-up:

$$\begin{array}{rll} m \models \varphi \wedge \psi & \text{iff} & m \models \varphi \text{ and } m \models \psi \\ m \models \varphi \vee \psi & \text{iff} & m \models \varphi \text{ or } m \models \psi \\ m \models \varphi \to \psi & \text{iff} & \forall n \sqsubseteq m.\, n \models \varphi \ \text{ implies } \ n \models \psi \end{array}$$

We must also handle the units of $\wedge$ and $\vee$, $\top$ and $\bot$, respectively:

$$\begin{array}{ll} m \models \top & \text{always} \\ m \models \bot & \text{never} \end{array}$$

While the clause for $\top$ is straightforward, we shall see later that that for $\perp$ is somewhat problematic.

All propositions here satisfy

Kripke Monotonicity (K): $m \models \varphi$ and $n \sqsubseteq m$ implies $n \models \varphi$,

as long as we assume this condition of propositional variables.

Given these definitions, we can define a semantic notion of logical consequence. Given a monoid as above and an interpretation of propositional variables compatible with Kripke Monotonicity, we say that

$\varphi \models \psi$ holds in the model iff for all $m \in M$, if $m \models \varphi$, then $m \models \psi$.

Lemma 1 (soundness) *If $\varphi \vdash \psi$ then $\varphi \models \psi$ in any monoid model.* $\square$

Soundness can also be stated in algebraic terms, by saying that the the collection of downwards-closed subsets of a preordered commutative monoid forms a **BI**-algebra. The semantic clauses just given specify the algebra structure. Note that there is nothing essential about intuitionistic logic here. In the special case in which the preorder is an equivalence relation, the semantics will validate the law of the excluded middle, so we get a Boolean **BI**-algebra.

The possible worlds semantics gives a large number of models of **BI**, for the simple reason that there are so very many (preordered) commutative monoids. This brings up a curious historical fact. In many presentations of systems of substructural logic, this or a similar semantics is altered, typically by imposing additional conditions, with the effect of precluding the existence of the additive implication $\rightarrow$ (be it intuitionistic or classical). The reason, so it seems [26], is that if one omits Weakening from standard sequent calculus then the law

$$\varphi \wedge (\psi \vee \xi) \dashv\vdash (\varphi \vee \psi) \wedge (\varphi \vee \xi)$$

of distribution is lost (distribution is a consequence of having a full strength $\rightarrow$).

However, this seems to get things the wrong way around: a simple semantics is altered to match a somewhat singular choice in the proof theory. The result is a logic in which it is very difficult to read the additive connectives in a simple way — where $\wedge$ means "and" and $\vee$ means "or": these lead to distribution.

Independently of these general arguments, we stress a practical point: to deny $\rightarrow$ is to deny access to the structure of a host of simple, naturally occurring, models. Of course, if none of these models were interesting the price would not be so great; this brings us back to what we call the *resource interpretation* of the connectives.

To repeat, in the resource interpretation we think of a formula as making a declarative statement about some state-of-affairs, but the truth of it is to be judged relative to access to available resources. Consider $\varphi * \psi$. We read it informally as follows:

> $\varphi * \psi$ is true just if the current resource can be decomposed into constituents in such a way as to make φ true of one constituent and ψ true of the other.

Similarly, we read $\varphi \mathrel{-\!\!*} \psi$ as follows:

> $\varphi \mathrel{-\!\!*} \psi$ is true just if, whenever we are given resources that make φ true, combining with what we already have, ψ will then also be made true.

This kind of reading also works for the additive connnectives:

> $\varphi \rightarrow \psi$ is true just if any consistent resource that makes φ true also makes ψ true.

The fact that the reading works for the full-strength additive implication, as well as conjunction and disjunction, is significant, since it is the extension of a reading of multiplicatives to other connectives that is often problematic in substructural logics. (These readings are similar to those given in the cost example but do not have to do with cost *per se*.)

We conclude by remarking that an alternative presentation of the semantics of substructural connectives is both possible and commonplace in relevant logic [7]. Briefly, our use of a monoidal product $\cdot$ together with an order $\sqsubseteq$ can be replaced by a ternary relation R on a set of worlds, so that, for example, the forcing relation for $\mathrel{-\!\!*}$ is rendered as

$$l \models \varphi \mathrel{-\!\!*} \psi \quad \text{iff} \quad \text{for all } m, n \in M \text{ such that } R(l, m, n), \text{ if } m \models \varphi, \text{ then } n \models \psi.$$

As we remarked before, this point of view is consistent with Day's [19] use of *premonoidal categories* in his treatment of tensor products on functor cateogries.

3.3 Example: Petri Nets

Petri nets provide a basic, concrete, model of computation, which fits well with the resource interpretation. A central tenet of the net theory is that resource is distributed throughout a net, in the form of *tokens* that reside in *places*. A distribution of tokens is called a marking; a net evolves according to local rules which show how to go from one marking to another. As in [27], we consider a basic notion of net which does not have capacities.

Formally, a net $\mathcal{N} = (P, T, pre, post)$ consists of sets P and T of places and transitions and two functions $pre, post : T \to \mathcal{M}$, from transitions to markings, where a marking is a finite multiset of places and $\mathcal{M}$ denotes the set of all markings. A marking can be regarded as a function $M : P \to N$ from places to natural numbers that is zero on all but finitely many places. Addition of markings is given by $(M + N)p = Mp + Np$. We let $[-]$ denote the empty marking.

There are several ways that nets can be used to provide a model of **BI**. One way internalizes the reachability relation on markings, by conflating it with the intuitionistic ordering in the model. If M and N are markings, then define

$$M \Rightarrow N \quad \text{iff} \quad \text{there are} \quad t, M' \quad \text{such that} \quad M = pre(t) + M' \quad \text{and} \quad N = post(t) + M' .$$

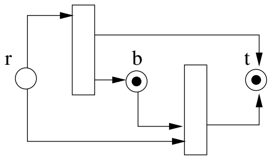
We can then define a preorder on markings by

$$M \sqsubseteq N \quad \text{iff} \quad \text{there are} \quad M_1, \ldots, M_n \quad \text{such that} \quad M = M_1 \Rightarrow \cdots \Rightarrow M_n = N$$

Then $(\mathcal{M}, [-], +, \sqsubseteq)$ is a preordered commutative monoid and so this gives us an interpretation of all the connectives.

Now, this model is just the Petri net semantics of linear logic described by Engberg and Winskell, except that they did not include $\to$. This omission seems strange in retrospect, given that it exists naturally in the model. Admitting it enables some of the discrepancies between model and logic observed by Engberg and Winskell to be avoided. These include the need to state an axiom for distrubition of $\vee$ over $\wedge$, which is implied by the more primitive rules for $\to$, as well as the ability to state negative properties of nets using $\neg\varphi = \varphi \to \bot$.

A basic example is mutual exclusion, where we say that two places cannot be marked at the same time. To see how this works, consider the following net, which represents processes either producing an item to a buffer or consuming an item from the buffer:



r and t denote ready processes and terminated processes, respectively and b represents a buffer whose tokens are items produced. Then we can say that a process is not both ready and terminated using $\neg(\mathrm{r} * \mathrm{t} * \top)$. Using $-\!\!*$, we can further say that a process is not both ready and terminated in any marking reachable from a given marking $M_0 : M_0 -\!\!* \neg(\mathrm{r} * \mathrm{t} * \top)$. Note the role of $\top$ in $\mathrm{r} * \mathrm{t} * \top$. It enables the state, at a given time, to be partitioned into three parts where r is true in one, t in another, and where the third part is arbitrary.

Thus far we have presented Engberg and Winskell's model, and made the simple point that $\to$ exists naturally in it. We move on to make a criticism of the model, and offer a few speculative remarks on further possibilities.

The criticism has to do with the internalization of the the reachability relation of nets: It can be argued that this is an unnatural choice. In fact, in describing the mutual exclusion example we skated over two complications. First, the interpretation of a place as a proposition, say r, cannot be just a singleton marking: rather it has to be the set of all markings that can reach the singleton marking of interest, from the past. Second, if we unwind the formula $\neg(\mathrm{r} * \mathrm{t} * \top)$ then semantically there is a complex back-and-forth between future and past; this seems to hamper rather than help understanding.

Also, in this semantics, all expressible properties are closed under (backwards) reachability, and this excludes many properties of interest.

An alternative model is given by the monoid $(\mathcal{M}, [-], +, =)$, where we do not include reachability. This results in a model of Boolean **BI**, which allows net properties to be expressed directly. For example, $\neg(\mathrm{r} * \mathrm{t} * \top)$ is true of a marking just when r and t are not both marked; compare to the back-and-forth in the intuitionistic version. This straightforward use of classical logic is similar to the approach taken by Reisig [28], but extended with multiplicatives.

Since we have removed the reachability relation from the semantics of **BI** formulæ, we have to incorporate net dynamics by other means. There are several standard devices for doing this, such as modal or temporal logics. For example, a modal operator for transitions can be defined as follows:

$$M \models \langle t \rangle \varphi \quad \text{iff} \quad \exists M'.\, M = pre(t) + M' \text{ and } post(t) + M' \models \varphi$$

As an example of a specification using this modality,

$$\langle t_1 \rangle \top \ \wedge \ \langle t_2 \rangle \top \ \wedge \ \neg(\langle t_1 \rangle \top \ * \ \langle t_2 \rangle \top)$$

says that t_1 and t_2 are currently enabled, but in conflict, so that one of them can fire but not both. We can also describe a valid inference rule that expresses the local nature of transitions:

$$\frac{P \to \langle t \rangle P'}{P * Q \to \langle t \rangle (P' * Q)}$$

This could be taken further using a modality for concurrent transitions. The main point, however, is that because the meaning of a formula is confined to certain resources, in this case a marking, we can use $*$ to set its effects apart from other local states and transitions.

The combination of classical logic to describe properties of markings, multipicatives to describe separation, and modalities for firing is much easier to understand than the intuitionistic model, which attempts to force dynamics into the basic model. Stated more positively, the model exhibits a *separation of concerns* between different aspects of resource that is often desirable. That is, it is possible to use standard techniques (temporal logics, pre- and post-conditions) to account for the consumption, or dynamics, of resources, together with a substructural view which in many examples is primarily spatial.

3.4 CCS-like Models

Process calculi typically contain an operation for the parallel composition of processes which is associative and commutative, with a unit. In this section, we describe how to build models based on a fragment of CCS.

We consider a small fragment, as follows:

P	$::=$	0	Null Process
	$\mid$	$P \mid P$	Parallel Composition
	$\mid$	$P + P$	Nondeterministic Composition
	$\mid$	$\alpha.P$	Prefixing
	$\mid$	$\cdots$	

Here, α ranges over actions, which often includes a silent action, names and co-names.

Associated with process terms are a number of equivalence relations $P \equiv P'$, the most significant of which are strong and weak bisimulation. It is also possible to take a very fine equivalence, such as structural congruence. We do that here, simply taking $\equiv$ to be the equivalence relation generated by commutative monoid equations for 0 and $\mid$, and for 0 and $+$. Since $\mid$ is commutative and associative with respect to $\equiv$ it immediately gives rise to a monoid model $(\mathcal{P}, 0, \mid, \equiv)$, where the process terms are possible worlds. Because two processes are related just when they are equivalent, this is a model of Boolean BI.

Just as with the second Petri net model, this semantics does not, by itself, account for the dynamics of processes. For this, we could add a Hennessy-Milner modality

$$P \models [a]\varphi \quad \text{iff} \quad \text{for all } Q, \text{if } P \xrightarrow{a} Q, \text{then } Q \models \varphi.$$

where $\xrightarrow{a}$ is a relation which says that "P can do an a and evolve to Q."

This line was followed by Dam in his thesis [29] with only limited success. From our point of view, the most curious point was that he imposed additional conditions on the models, which had the effect of removing the additive implication. This seems unfortunate, as the additive connectives naturally exist and a much simpler model is obtained by retaining them. One might speculate that a better logic would obtain if one were to keep the (Boolean) additives. In fact, this is part of what Cardelli and Gordon do, in their logic of ambients [23].

4 Completeness and Incompleteness

The monoid semantics gives us a way to construct **BI**-algebras but the algebras obtained are very special. Although it gives a limited class of models, Kripke's semantics is still complete for intuitionistic logic. So it is natural to ask: what is the status of the elementary monoid models with respect to **BI** ?

The key issue is the handling of *inconsistency* in the presence of multiplicative connectives. The propositions p and $\mathrm{p} \mathrel{-\!\!*} \bot$ are both consistent yet the proposition $\mathrm{p} * (\mathrm{p} \mathrel{-\!\!*} \bot)$ is inconsistent since

$$\mathrm{p} * (\mathrm{p} \mathrel{-\!\!*} \bot) \vdash \bot.$$

This is not a problem in and of itself, but the fact that $*$ does not preserve consistency, *together with the treatment of* $\bot$ *in the elementary semantics*, leads to incompleteness.

Proposition 2 (elementary incompleteness) $(\mathrm{p} \mathrel{-\!\!*} \bot) \to \bot \wedge (\mathrm{q} \mathrel{-\!\!*} \bot) \to \bot \models (\mathrm{p}*\mathrm{q} \mathrel{-\!\!*} \bot) \to \bot$ *in the elementary monoid semantics but* $(\mathrm{p} \mathrel{-\!\!*} \bot) \to \bot \ \wedge\ (\mathrm{q} \mathrel{-\!\!*} \bot) \to \bot \not\vdash (\mathrm{p} * \mathrm{q} \mathrel{-\!\!*} \bot) \to \bot$ *in* **BI**'*s calculus.*

Proof The key to showing incompleteness is that the formula $(\varphi \mathrel{-\!\!*} \bot) \to \bot$ expresses consistency of φ in the monoid semantics, in the following sense:

$m \models (\varphi \mathrel{-\!\!*} \bot) \to \bot$ holds iff there is an n such that $n \models \varphi$.

Now, we can use the totality of the monoid operation against itself.

To see why the semantic judgement in the proposition is true, given n where $n \models \mathrm{p}$ and m where $m \models \mathrm{q}$, we have that $n \cdot m \models \mathrm{p} * \mathrm{q}$ and, because of the existential formula characterizing $(\mathrm{p} \mathrel{-\!\!*} 0) \to 0$, this is enough to give us the judgement.

The unprovability of the syntactic judgement is easy to establish via the cut-elimination theorem for **BI**'s sequent calculus [2, 3]. (Also, at the end of § 6.3, we give an explicit counter-model.) □

A more conceptual, partial explanation of this incompleteness can be seen by considering where a standard completeness argument breaks down. In this (which is essentially a Yoneda lemma argument), we use the propositions of **BI** to build a term model. Formally, the term monoid has the set of formulæ as its underlying set and the order and monoid structure are given by

- $\varphi \sqsubseteq \psi$ iff $\varphi \vdash \psi$,
- $\varphi \cdot \psi \ =\ \varphi * \psi$ and
- $e\ =\ I$.

Then the main subsidiary lemma is

$\varphi \vdash \psi$ is derivable iff $\varphi \models \psi$ in the term model

This lemma is established by a routine induction on ψ, but there is a sticking point: the proof breaks down when we encounter $\vee$ or $\bot$. For example, from $\varphi \vdash \psi_0 \vee \psi_1$ it does not follow that $\varphi \vdash \psi_0$ or $\varphi \vdash \psi_1$, as would be needed for the result: take $\varphi = \psi_0 \vee \psi_1$. Similarly, the monoid semantics would require that $\varphi \vdash \bot$ never holds: but this is not the case when $\varphi = \bot$.

However, the proof based on the Yoneda lemma does go through for the $(\bot, \vee)$-free fragment, so we may conclude the following:

Proposition 3 (completeness for $(\bot, \vee)$-free fragment) *If φ and ψ are $(\bot, \vee)$-free formulæ then $\varphi \vdash \psi$ is derivable iff $\varphi \models \psi$ in all monoid models.* □

The absence of $\vee$ is not important, however. The failure of the argument of $\vee$ represents a failure of the easy proof, based on the Yoneda lemma, rather than the failure of completeness. In [3], it is shown that this result can be extended to the $\bot$-free fragment, using an argument (beyond our present scope, but see § 6.2) based on the construction of *prime bunches*. Proposition 2 shows that the restriction on $\bot$ cannot be removed.

We emphasise also that the incompleteness of the elementary monoid semantics arises from the *interaction* between the two implications and inconsistency, $\bot$: The usual completeness argument for Kripke semantics [30] goes through for the intuitionistic fragment (lacking $*$, I, $-\!\!*$) without difficulty.

So, what are we to make of our completeness and incompleteness results so far ? The answer lies in the internalization of inconsistency by the semantics. Consider that the (complete) algebraic models in § 3.1 or, more generally, the categorical semantics of proofs in [1, 3] include representatives for inconsistency (the initial object, **0**, which interprets $\bot$). The elementary forcing semantics, in contrast, can handle inconsistency only by denying the existence of a world at which $\bot$ is forced. Completeness for a monoid-based forcing semantics can be achieved in settings in which internal represenitves for inconsistency are available. We develop such a semantics in §§ 6 and 6.3.

It is worth pausing to take stock. We began in the introduction by presenting an informal cost model and we extrapolated from it to arrive at the elementary monoid semantics. We have since seen several example models, but also a technical problem in the elementary semantics: the treatment of $\bot$ is incomplete.

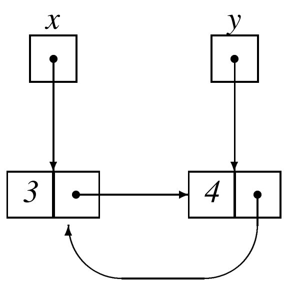
At this point we are going to repeat the same tack; that is, we will consider a concrete model which expresses some ideas about resource and then present a more general context in terms of which it can be understood.

We should begin this second tack by reconsidering the assumptions that lead to the elementary monoid semantics. We have discussed internalizing $\bot$. In terms of resources, a way to see the problem is to trace it to the assumption that the combining operation $\cdot$ is always defined; but why, in terms of resources, should it be ? We might expect, in fact, to only be able to combine resources when they are consistent. A good example of this is provided by portions of the computer store: if m describes one portion and n another, then we might require that they describe disjoint areas of storage in order for them to be combined. We now present a specific model based on this idea, and then continue with our theoretical development.

5 Example: A Pointer Model

In this section, we present a model in which "resource" corresponds to "portion of the computer store". The store is made up of cons cells, which can have basic data (such as integers) in their components, or pointers to other cons cells. The model presented in this section is from work on using **BI** to reason about pointers [31], which builds on work of Reynolds [32]. A related example, presented from the point of view of a dependently-typed λ-calculus which is intimately related to **BI**, can be found in [16].

The inclusion of pointers brings out several issues, most importantly *sharing*. That is, data structures are often constructed so that there are two or more pointers to the same cell, as happens when considering graphs or circular or doubly-linked lists. When this happens, there are multiple ways to refer to the same cell, or in short, there is *aliasing*. For example, if we use the notations $x.1$ and $x.2$ to refer to the first and second components of a cons cell then x, $y.2$ and $x.2.2$ are all aliases in the situation represented by the following box-and-pointer diagram:



Since aliasing is extremely delicate, we take more time over this model than for the others. Sharing was not present in the Petri net, process or cost models: they only exhibit combining.

Formally, the worlds in this model are heaps $h \in H$, which are thought of as collections of cons cells in storage.

$$\begin{array}{rcl} Val & = & Int \cup \{nil\} \cup Loc \\ H & = & Loc \rightharpoonup_{fin} Val \times Val \end{array}$$

Here, $Loc = \{\ell, ...\}$ is an infinite set of locations and $\rightharpoonup_{fin}$ is for finite partial functions. Each cell in memory is identified by a location and when $h(\ell) = (a, b)$ this represents a situation in which ℓ has a in its first component and b in its second. When $h(\ell)$ is undedined this represents a situation where there is no cell in the heap corresponding to ℓ.

We use a combining operation on heaps that is partial:

- $h \cdot h'$ denotes the union of disjoint heaps (*i.e.*, the union of functions with disjoint domains); e is the empty heap. When the domains of h and h' overlap, $h \cdot h'$ is undefined.

The order we consider at this point is discrete: the equality relation on H, and the clauses for the additive connectives remain as in the elementary monoid semantics. This gives us a Boolean **BI**-algebra, where the Boolean algebra part is just the set of subsets of H. (An alternative, intuitionistic, model is also of interest: it works by taking the relation $h \sqsubseteq h'$ between worlds to be graph superset of partial functions [32].)

The adjustments we must make are to the semantics of multiplicatives; we include definedness conditions:

$$h \models \varphi * \psi \quad \text{iff} \quad \text{there are } h_0, h_1 \text{ such that} (h_0 \cdot h_1) \downarrow \text{ and } h_0 \cdot h_1 = h \text{ and } h_0 \models \varphi \text{ and } h_1 \models \psi$$

$$h \models \varphi \mathrel{-\!\!*} \psi \quad \text{iff} \quad \text{for all } h', \text{ if } (h \cdot h') \downarrow \text{ and } h' \models \varphi \text{ then } h \cdot h' \models \psi$$

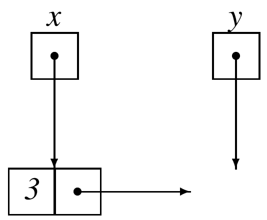
In order to describe atomic propositions, we assume a function $s : Var \to Val$ where $Var = \{x, y, ...\}$ is a set of variables. Since we consider s to be given once and for all, we are technically remaining in a propositional setup; this obviously paves the way, however, to a consideration of quantifiers.

The basic proposition is the points-to relation, which has the form $x \mapsto E, F$, where E and F range over variables, integers and nil.

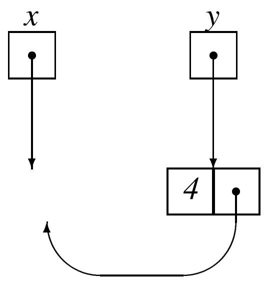
$$h \models x \mapsto E, F \quad \text{iff} \quad \{sx\} = dom(h) \text{ and } h(sx) = \langle [\![E]\!]s, [\![F]\!]s \rangle,$$

where $[\![E]\!]s$ gives the value of E in s. Notice the exact nature of this interpretation, where the domain of h is required to be a singleton: $x \mapsto E, F$ means that x points to E, F in the current heap, and also that x is the only cell in the current heap.

As a first example in this model, the formula $(x \mapsto 3, y) * (y \mapsto 4, x)$ corresponds to the box-and-pointer diagram pictured earlier. To relate this picture to the formal definition, if the formula is true at a heap h, then we must have that sx and sy are locations, by the definition of $\mapsto$, and that they are distinct, by the definition of $*$. For, $*$ splits h into two subheaps, one where sx is the only defined location and the other where sy is defined. Notice the importance of dangling pointers here: the picture corresponding to the left conjunct is



while that for the right is

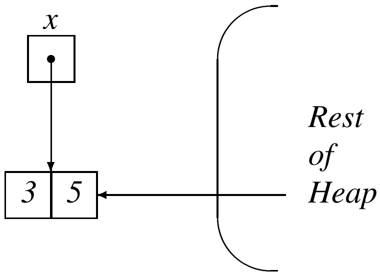


Notice that in each subheap we have a dangling pointer, which is a location not in the domain of the heap.

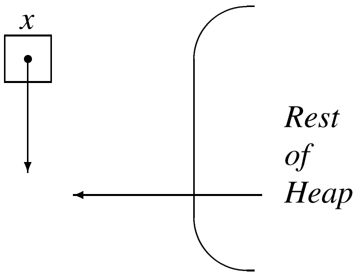
While $*$ is about separation, the implication $\mathrel{-\!\!*}$ can be used to describe new, or fresh, pieces of storage. These two connectives interact in an interesting way: The formula

$$(x \mapsto 3, 5) \; * \; ((x \mapsto 7, 5) \mathrel{-\!\!*} P).$$

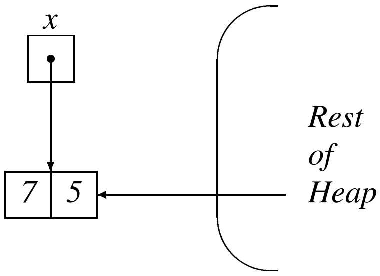
says that $(x \mapsto 3, 5)$ is true in the current heap, but also that if we update the first component to 7 then P will be true. To see why, first note that the semantics of $*$ splits the heap, say,



into two portions, one where $(x \mapsto 3, 5)$ and a second heap where the location denoted by x is dangling:



We have included a dangling pointer out of the rest of the heap here to emphasize that the location might be referenced from within a heap cell, as well as from x. Because the association $(x \mapsto 3, 5)$ has been, in a sense, retracted by deleting the association from the heap in the right conjunct, this frees $-\!\!*$ to extend the second heap with a different cons cell. The semantics of $-\!\!*$ then ensures that P must be true when this second heap is extended with a new binding of location to contents that makes $(x \mapsto 7, 5)$ true.



So, the intuitive description in terms of updating follows from several steps in the semantics, which add up to "update as deletion followed by extension".

This idea can be used to formulate the weakest precondition for assignment statements $x.1 := y$ and $x.2 := y$ that alter the first or second component of a cons cell in the heap. Here is the Hoare triple axiom for $x.1$, together with an axiom for an operation for allocating a cons cell, and an axiom for an operation that deallocates a cell. (The axioms require the evident extension of the model to include quantifiers.)

$$\begin{array}{l} \{\exists wz.\,(x \mapsto w, z) * ((x \mapsto y, z) -\!\!* P)\} \\ x.1 := y \\ \{P\} \end{array}$$

$$\begin{array}{l} \{\forall x.\,(x \mapsto y, z) -\!\!* P\} \\ x := \mathit{cons}(y, z) \\ \{P\} \end{array}$$

$$\begin{array}{l} \{P * \exists yz.\,(x \mapsto y, z)\} \\ \mathit{dispose}(x) \\ \{P\} \end{array}$$

The *Dispose* axiom is particularly interesting. It says that that you simply shouldn't depend on what contents the disposed location might or might not have in the post-condition.

Because we have considered a trivial ordering, the surrounding connectives are interpreted in a pointwise, thus classical, fashion. For instance, if z denotes a cell then $\neg(\exists xy.\,(x \mapsto y, z) * \top)$ says that there is no cell in the heap which has a pointer to z in its second component.

Earlier, we referred to the idea that a judgement $m \models \varphi$ may be though of in a "local" way, where the resources referred to by φ are confined to those in m. To see how this works in the pointer model, consider the formula $(x \mapsto y, z) * (z \mapsto w, nil)$. It may be that y and w are pointers, but the truth of this statement is judged in a way that ignores what they point to. The reason is that the semantics of $*$ and $\mapsto$ together ensures that the truth of the formula can be established only at a heap where there are precisely two defined locations.

A consequence of this local nature is that specifications can be given in a way that ignores cells that are irrelevant to the correct operating of a program. For example, the triple

$$\{(x \mapsto a, y) * (y \mapsto c, z)\}\ C\ \{\exists t.\,(x \mapsto a, t) * (t \mapsto b, y) * (y \mapsto c, z)\}$$

specifies part of a program for instering an element into a linked list, in a way that only mentions the cells that are relevant at the insert stage. In [31] an axiom for inferring frame axioms is given, which enables local specifications of this sort to interact with more global contexts.

The examples we have given in this section were chosen to give a feeling for the spatial way of understanding the connectives in the pointer model. We refer to [32, 31, 33] for more information. In particular, [33] contains a number of worked verifications, as well as a generalization of the model of this section to allow for pointer arithmetic.

We conclude by remarking that the semantic structure of this model is incompatible with the formal system of linear logic. To see this, consider that $\varphi \multimap \psi \models \varphi \to \psi$ always holds in linear logic, using the decomposition $\varphi \to \psi\ = !\varphi \multimap \psi$ and the rule of Dereliction for !. However, here

$$(x \mapsto 1, 2) \mathrel{-\!\!*} \bot \not\models (x \mapsto 1, 2) \to \bot$$

because the antecedent can hold in a heap where $x \mapsto 1, 2$ while the consequent cannot. This shows that there can be no ! which decomposes $\varphi \to \psi$ into $!\varphi \mathrel{-\!\!*} \psi$ in this model.

We make this point not in criticism of linear logic but merely in support of our contention that there are interesting and naturally occurring models in which both $\mathrel{-\!\!*}$ and $\to$ exist independently. It is natural to want to have access to the structure of these models, independently of abstract arguments in favour of one logic or another.

6 Topological Semantics

There are several mathematical ways to incorporate the kind of partiality found in the pointer model, including taking a partial operation as primitive and taking a ternary relation semantics as primitive. However, rather than try to fiddle with the semantics, it seems advisable at this point to take a more mathematical turn, and observe some of the lessons learnt in the model theory of intuitionistic logic (see [21]). Briefly, Kripke models are a special form of topological model, where the open sets are the down-closed subsets of a preorder. And toplogical models are in turn a special kind of Grothendieck sheaf model. Pragmatically speaking, since topological ideas give rise to many interesting models of intuitionistic logic, we would like to have access to these in the model theory of **BI**.

While it may well be possible to give sound and complete models of **BI** using methods which are mathematically less sophisticated, we believe the topological and sheaf-theoretic methods which we adopt in this section, as in [1, 2, 15, 3], give the appropriate level of clarity and elegance. That being said, we will not attempt to justify the definitions in the Grothendieck semantics by appeal to resource. But we do claim that the definitions are consistent with (which is not to say implied by) our previous discussion.

So, returning to our theoretical development, we describe three classes of models based on topological structures:

- *Open topological monoids.*
- *Sheaves on open topological monoids.*
- *Grothendieck sheaves on preordered monoids.*

Each of these classes of topological models yields a (a soundness and) completeness theorem for **BI**. The unifying feature of these models in respect of completeness is their internalization of inconsistency via their semantics for $\bot$.

In keeping with our gradual style, we present the first two briefly, as stepping stones on our way to our final notion of model, for which we present a detailed proof of soundness and completeness. Our main addition in each case will be to include an appropriate continuity condition on the monoid operation in question.

6.1 Open Topological Monoids

A (commutative) topological monoid is a (commutative) monoid in the category **Top** of topological spaces and continuous maps between them, *i.e.*, a topological space $\mathcal{X}$, with open sets $\mathcal{O}(\mathcal{X})$, together with two arrows, a tensor product $* : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X}$ and its unit $e : 1 \longrightarrow \mathcal{X}$ such that the usual monoidal diagrams commute [34].

We need to interpret a formula $\varphi * \psi$ as the tensor product, $U * V$ of the interpretations, respectively U and V, of φ and ψ. The tensor product of two open sets is not necessarily open, however. Consequently, we must require that the monoidal structure be defined by open maps, *i.e.*, which map open sets to open sets.

An *open* topological monoid is one in which the maps $*$ and e, which define the monoidal structure are open.

Lemma 4 (distributivity) *Let $(\mathcal{X}, *, e)$ be a topological monoid. Then, for all open sets U, V_i, $i \in \mathcal{I}$, where $\mathcal{I}$ is some indexing set,*

$$U * (\bigcup_i V_i) = \bigcup_i (U * V_i).$$

Proof $z \in U * (\bigcup_i V_i)$ iff there exist $x \in U$ and $y_j \in V_j$, for some j, such that $z = x * y_j$ iff $z \in \bigcup_i (U * V_i)$. □

The interpretation of **BI** in an open topological monoid now follows exactly as for the interpretation of intuitionistic logic in a topological space, *i.e.*, with $[\![\bot]\!] = \emptyset$, with the addition of the following:

$$\begin{array}{rcl} [\![\varphi * \psi]\!] & = & [\![\varphi]\!] * [\![\psi]\!] \\ [\![I]\!] & = & e(1) \end{array}$$

and if $[\![\varphi]\!] = U$ and $[\![\psi]\!] = V$, then

$$[\![\varphi \mathrel{-\!\!*} \psi]\!] \quad = \quad \bigcup_{i \in \mathcal{I}} W_i,$$

where each W_i is such that $W_i * U \subseteq V$. This interpretation is well-defined:

Lemma 5 (multiplicative function space) $[\![\varphi \mathrel{-\!\!*} \psi]\!] * [\![\varphi]\!] \subseteq [\![\psi]\!]$.

Proof We have $\bigcup_{i \in \mathcal{I}} (W_i * U) \subseteq V$, so that $(\bigcup_{i \in \mathcal{I}} W_i) * U \subseteq V$, by distributivity. □

We can obtain soundness and completeness for these models just as for **BI**-algebras. We conjecture that examples can be found in Euclidian spaces.

6.2 Sheaf-theoretic Models

An alternative way to give a topological semantics to **BI**, instead of the algebraic treatment in § 6.1, is to give a forcing semantics in the category of sheaves over a topological monoid.

We start with a commutative open topological monoid, $\mathcal{X} = (\mathcal{X}, *, e)$. The symmetric monoidal structure of a (commutative) topological monoid, $\mathcal{X}$, gives rise, via Day's construction of a tensor product [19, 1], to a symmetric monoidal closed structure on the category $\mathbf{Sh}(\mathcal{X})$ of sheaves on $\mathcal{X}$ [3]. Let L be a set of propositional letters. Then a *topological Kripke* **BI***-model* is, subject to some conditions, a triple $\langle \mathbf{Sh}(\mathcal{X}), \models, [\![-]\!] \rangle$, satisfying the constraints in Table 1. It is a routine matter to check that this definition is consistent with the usual presentation of the sheaf-theoretic semantics of intuitionistic logic [35].

We do not give detailed proofs of the soundness and completeness of **BI** for topological Kripke **BI**-models, preferring to give these results to the more general setting of Grothendieck sheaves in § 6.3. The details of these results can be found in [3], for both propositional **BI** and predicate **BI** [2]. However, a few remarks, will be informative.

We sketch the construction of a term model, which would be the basis of a completeness proof. The key idea we require is that of a bunch [7, 1]. Briefly, **BI**'s bunches are contexts which have two symmetric monoidal combining

$U \models \mathrm{p}$	iff	$[\![\mathrm{p}]\!](U) \neq \emptyset$, for $\mathrm{p} \in L$
$U \models \varphi * \psi$	iff	for some $V, V' \in \mathcal{O}(\mathcal{X}), U \subseteq V \cdot V'$ and $V \models \varphi$ and $V' \models \psi$
$U \models \varphi \mathrel{-\!\!*} \psi$	iff	for all $V \in \mathcal{O}(\mathcal{X}), V \models \varphi$ implies $U \cdot V \models \psi$
$U \models \varphi \wedge \psi$	iff	$U \models \varphi$ and $U \models \psi$
$U \models \varphi \vee \psi$	iff	for some $V, V' \in \mathcal{O}(\mathcal{X})$ such that $U = V \cup V'$, $V \models \varphi$ and $V' \models \psi$
$U \models \varphi \to \psi$	iff	for all $V \subseteq U$, $V \models \varphi$ implies $V \models \psi$
$U \models \top$	for all	$U \in \mathcal{O}(X)$
$U \models I$	iff	$U \subseteq I$
$U \models \perp$	iff	$U = \emptyset$

Table 1: Semantics in Sheaves

operations, ",", which corresponds to $*$ and ";", which corresponds to $\wedge$ and is therefore cartesian. Thus bunches are structured as trees, with formulæ at the leaves and combining operations at the internal nodes. It follows that corresponding to each bunch Γ there is a logically equivalent formula φ_Γ.

We can define a term topological Kripke **BI**-model, in which we suppress the routine definition of $[\![-]\!]$, as follows:

- $\mid \mathcal{X} \mid$ is $B/\dashv\vdash$, where B is the set of sets of consistent bunches and where $\dashv\vdash$ is the evident equality generated by derivability, *i.e.*, if S and S' are sets of consistent bunches, then $S \vdash S'$ iff, for any $\Gamma \in S$, there exists $\Gamma' \in S'$ such that $\varphi_\Gamma \vdash \varphi_{\Gamma'}$;
- Open sets are elements of $\mathcal{X}$ closed under *prime evaluation* of bunches. The prime evaluation, $\lceil \Gamma \rceil$, of a bunch Γ is constructed as follows:
  - Close under consequences generated by the propositions in Γ. For example, closing $\Gamma = \Gamma(\varphi, \varphi \mathrel{-\!\!*} \psi)$ under consequences requires evaluating the bunch to $\Gamma(\psi)$, and closing $\Gamma = \Gamma(\varphi * \psi)$ under consequences requires evaluating such a bunch to $\Gamma(\varphi, \psi)$. Let $\overset{\leadsto}{\Gamma}$ denote the result of all such evaluations of a bunch Γ (see [3] for the details);
  - Extend Γ, using ";", with the bunch $\overset{\leadsto}{\Gamma}$, to get $\lceil \Gamma \rceil = \Gamma\,;\,\overset{\leadsto}{\Gamma}$. The bunch $\lceil \Gamma \rceil$ is such that if $\Gamma \vdash \varphi$, then $\lceil \Gamma \rceil \vdash \varphi$, if $\lceil \Gamma \rceil \vdash \varphi$, then $\lceil \Gamma \rceil = \lceil \Gamma \rceil(\varphi)$, and has the disjunction property.

  So, for any open set O, if $S \in O$ is a set of bunches and $\Gamma \in S$, then $\lceil \Gamma \rceil \in S$.

  Note that prime evaluation generates a set of bunches;
- The monoid operation, $\cdot$, is given by the consistent prime evaluation of the combination of bunches using the comma, ",": $\Gamma * \Delta \cong \lceil \Gamma\,,\,\Delta \rceil$, where $\cong$ denotes isomorphism of labelled trees, so that

  $$\{\Gamma_1, \ldots, \Gamma_m\} \cdot \{\Delta_1, \ldots, \Delta_n\} = \begin{array}{lllllll} \{ & \Gamma_1 * \Delta_1 & , & \Gamma_1 * \Delta_2 & , & \ldots & \\ & \Gamma_2 * \Delta_1 & , & \Gamma_2 * \Delta_2 & , & \ldots & \\ & \vdots & , & \vdots & , & \ldots & \} \end{array} \setminus \perp(\Gamma, \Delta)$$

  where $\perp(\Gamma, \Delta) = \{\Gamma_i * \Delta_j \mid \Gamma_i * \Delta_j \vdash \perp\}$;
- The unit, e is given by $\{\emptyset_m\}$, where $\emptyset_m$ is the unit of ",";

- $[\![\varphi]\!](\Gamma) = \{\, \Phi \mid \Phi$ is a proof of $\Gamma \vdash \varphi \,\}$. Here we intend a restriction of Φ to normal proofs in **BI**'s natural deduction system [1, 2, 3].

Notice that we remove the inconsistent bunches and that we will always be left with at least the empty set. This property of the term model, together with the appropriate modificaton of the forcing clause for $\bot$, yields completeness. To see this, consider the following example: Let $\mathrm{p} \in L$. Then both p and $\mathrm{p} \mathrel{-\!\!*} \bot$ are consistent bunches but their monoidal combination is not. However, it is easy to see that $\lceil \mathrm{p} \cdot \mathrm{p} \mathrel{-\!\!*} \bot \rceil = \emptyset$ and $\emptyset \models \bot$. How is this to be seen as being consistent with the elementary forcing semantics, in which $\bot$ is never forced ? The answer is simply that, in order for completeness to go through in the presence of inconsistency, we must use a setting in which "never" is part of the model: the empty set fills exactly this rôle: just as in the elementary monoid semantics, models inhabit functor categories $\mathbf{Set}^{\mathcal{M}^{op}}$ but for completeness with $\bot$, we refine this setting to that of sheaves (with the corresponding modification of the forcing relation) on an $\mathcal{M}$ which is a topological space.

An alternative construction of a term model would follow the pattern of Proposition 3 and use open sets of formulæ, closed under $\vdash$. However, in such a setting, lacking immmediate recourse to prime bunches, it is difficult to see how to handle disjunction well. In the more general setting of Grothendieck sheaves, developed in § 6.3 and in which we give detailed soundness and completeness proofs, disjunction is handled via the Grothendieck topology.

6.3 Grothendieck Sheaf-theoretic Models

In this section, we give a class of models which generalizes the ones we have so far described and in which we give a detailed proof of completeness. We work with Grothendieck topologies [35], the algebraic generalization of topological spaces, on preordered commutative monoids. This setting allows us to recover the appealing simplicity of the elementary preordered commutative monoid semantics whilst retaining the topological treatment of inconsistency, via the empty set, which gives rise to completeness in the presence of $\bot$. The connection between the two topological formulations is the usual one [35].

A *Grothendieck topological preordered commutative monoid*, in short, *a Grothendieck topological monoid* or *GTM*, is a preordered commutative monoid $\mathcal{M} = (M, \sqsubseteq, \cdot, e)$ together with a map $J : M \to \mathcal{P}(\mathcal{P}(M))$ satisfying some axioms. Such a J is usually called a *Grothendieck topology*.

There are two classes of axioms for GTM: four axioms for Grothendieck topology and one axiom for the "continuity" of "$\cdot$". The four axioms for Grothendieck topology are as follows:

- *Sieve*: for any $m \in M$ and $S \in J(m)$, "$S \sqsubseteq m$", *i.e.*, for any $m' \in S$, $m' \sqsubseteq m$;

- *Maximality*: for any n' such that $n' = n$, $\{n'\}$ is in $J(n)$ (If we were to work with partially-ordered monoids, then we should need the following maximality condition instead: for any $m \in M$, $\{m\} \in J(m)$.);

- *Stability*: for any $m, n \in M$ and $S \in J(m)$ such that $n \sqsubseteq m$, there exists $S' \in J(n)$ such that "$S' \sqsubseteq S$": for any $n' \in S'$, there exists $m' \in S$ such that $n' \sqsubseteq m'$; and

- *Transitivity*: for any $m \in M$, $S \in J(m)$ and $\{S_{m'} \in J(m')\}_{m' \in S}$, $\bigcup_{m' \in S} S_{m'} \in J(m)$.

The other axiom for the "continuity" of "$\cdot$" is:

- *Continuity*: for any $m, n \in M$ and $S \in J(m)$, "$S \cdot n \in J(m \cdot n)$", *i.e.*, $\{m' \cdot n \mid m' \in S\} \in J(m \cdot n)$.

In a preordered commutative monoid model, the interpretation of propositions satisfies the Kripke monotonicity condition:

- (K): for any $m, n \in M$ such that $n \sqsubseteq m$, if $m \models \varphi$, then $n \models \varphi$.

Let $\mathcal{M}$ be a GTM and let $\mathcal{P}(L)$ denote the collection of **BI** propositions over a language L of propositional letters. Then a *Grothendieck resource model* is, subject to some conditions, a triple $\langle \mathcal{M}, \models, [\![-]\!] \rangle$, satisfying the conditions Table 2 and not only (K) but also a sheaf-like condition, which is sometimes called *locality* [35]:

- (Sh): for any $m \in M$ and $S \in J(m)$, if "$S \models \varphi$", then $m \models \varphi$. More precisely, if for all $m' \in S$, $m' \models \varphi$, then $m \models \varphi$.

$m \models \mathrm{p}$	iff	$m \in [\![p]\!]$
$m \models \top$	iff	always
$m \models \varphi \wedge \psi$	iff	$m \models \varphi$ and $m \models \psi$
$m \models \varphi \to \psi$	iff	for any $n \sqsubseteq m$, if $n \models \varphi$, then $n \models \psi$
$m \models \varphi \vee \psi$	iff	there exists $S \in J(m)$ such that for any $m' \in S$, $m' \models \varphi$ or $m' \models \psi$
$m \models \bot$	iff	$\emptyset \in J(m)$
$m \models I$	iff	there exists $S \in J(m)$ such that for any $m' \in S$, $m' \sqsubseteq e$
$m \models \varphi * \psi$	iff	there exists $S \in J(m)$ such that for any $m' \in S$, there exist $n_\varphi, n_\psi \in M$ such that $m' \sqsubseteq n_\varphi \cdot n_\psi$, $n_\varphi \models \varphi$ and $n_\psi \models \psi$
$m \models \varphi \mathrel{-\!\!*} \psi$	iff	for any n, if $n \models \varphi$ then $n \cdot m \models \psi$

Table 2: Semantics in Grothendieck Sheaves

The semantics of propositions is parametrized by the interpretation $[\![-]\!]$ of atomic propositions which makes (K) and (Sh) hold for atomic propositions. Following [1], the semantics of **BI**'s propositions is given in Table 2 by the forcing relation $m \models \varphi$. The first two results give the well-definedness of the Grothendieck semantics.

Lemma 6 *Given an interpretation $[\![-]\!]$ which makes (K) and (Sh) hold for atomic propositions, (K) holds for the interpretation of any* **BI** *proposition χ.*

Proof For any $m, n \in M$ such that $n \sqsubseteq m$ and $m \models \chi$, we must show $n \models \chi$. The proof proceeds by the induction on the structure of the proposition χ. In most of the cases, the inductive step is immediate. We give just those cases which differ from the corresponding ones in the preordered commutative monoid semantics.

- $\chi = \varphi \vee \psi$: since $m \models \varphi \vee \psi$, there exists $S_m \in J(m)$ such that for all $m' \in S_m$, $m' \models \varphi$ or $m' \models \psi$. By the stability axiom, there exists $S_n \in J(n)$ such that for all $n' \in S_n$, $n' \sqsubseteq m'$ for some $m' \in S_m$. Then, by the induction hypothesis, $n' \models \varphi$ or $n' \models \psi$ for any $n' \in S_n$.
- $\chi = \bot$: since $m \models \bot$, $\emptyset \in J(m)$. By the stability axiom, $\emptyset \in J(n)$.
- $\chi = I$: since $m \models I$, there exists $S_m \in J(m)$ such that $m' \sqsubseteq e$ for all $m' \in S_m$. By the stability axiom, there is $S_n \in J(n)$ such that for any $n' \in S_n$, $n' \sqsubseteq m'$ for some $m' \in S_m$. Then, for any $n' \in S_n$, $n' \sqsubseteq e$.
- $\chi = \varphi * \psi$: since $m \models \varphi * \psi$, there exists $S_m \in J(m)$ such that for any $m' \in S_m$, there exist $a_{m'}, b_{m'}$ such that $m' \sqsubseteq a_{m'} \cdot b_{m'}$, $a_{m'} \models \varphi$ and $b_{m'} \models \psi$. By the stability axiom, there exists $S_n \in J(n)$ such that for any $n' \in S_n$, $n' \sqsubseteq m'$ for some $m' \in S_m$, from which $n' \sqsubseteq a_{m'} \cdot b_{m'}$ follows. Therefore, for any $n' \in S_n$, there exist $a_{n'}, b_{n'}$ such that $n \sqsubseteq a_{n'} \cdot b_{n'}$, $a_{n'} \models \varphi$ and $b_{n'} \models \psi$.

□

Lemma 7 *Given an intepretation $[\![-]\!]$ which makes (K) and (Sh) hold for atomic propositions, (Sh) holds for the interpretaion of any* **BI** *proposition χ.*

Proof For any $m \in M$ and $S \in J(m)$ such that $m' \models \chi$ for all $m' \in S$, we should show that $m \models \varphi$. We use the induction on the structure of χ.

- $\chi = \mathrm{p}$: this case follows from the assumptions about $[\![-]\!]$.
- $\chi = \top$: for any $n \in M$ including the case that $n = m$, $n \models \varphi$.
- $\chi = \varphi \wedge \psi$: for any $m' \in S$, $m' \models \varphi$ and $m' \models \psi$. By induction hypothesis, $m \models \varphi$ and $m \models \psi$.
- $\chi = \varphi \to \psi$: for any $n \sqsubseteq m$ such that $n \models \varphi$, by the stability axiom, there exists $S_n \in J(n)$ such that for any $n' \in S_n$, $n' \sqsubseteq m'$ for some $m' \in S$. Also, $n' \sqsubseteq n$ by the sieve condition on S_n. By (K), as stated in Lemma 6, $n' \models \varphi \to \psi$ and $n' \models \varphi$, which implies $n' \models \psi$. By the induction hypothesis, $n \models \psi$.
- $\chi = \varphi \vee \psi$: for any $m' \in S$, there exists $S_{m'} \in J(m')$ such that for any $u \in S_{m'}$, $u \models \varphi$ or $u \models \psi$. Let $S_m = \bigcup_{m' \in S} S_{m'}$. Then, $S_m \in J(m)$ because of the transitivity axiom. Moreover, for any $u \in S_m$, $u \models \varphi$ or $u \models \psi$. Therefore, $m \models \varphi \vee \psi$.
- $\chi = \bot$: for any $m' \in S$, $m' \models \bot$ and so $\emptyset \in J(m')$. Since $\emptyset = \bigcup_{m' \in S} \emptyset$ is in $J(m)$ by the transitivity axiom, $m \models \bot$.
- $\chi = I$: for any $m' \in S$, there exists $S_{m'} \in J(m')$ such that $u \sqsubseteq e$ for any $u \in S_{m'}$. Let $S_m = \bigcup_{m' \in S} S_{m'}$. Then $S_m \in J(m)$ by the transitivity axiom. Moreover, for any $u \in S_m$, $u \sqsubseteq e$. Therefore, $m \models I$.
- $\chi = \varphi * \psi$: for any $m' \in S$, there exists $S_{m'} \in J(m')$ such that for any $u \in S_{m'}$, there exist a_u, b_u such that $u \sqsubseteq a_u \cdot b_u$, $a_u \models A$ and $b_u \models B$. Let $S_m = \bigcup_{m' \in S} S_{m'}$. Then, by the transitivity axiom, $S_m \in J(m)$. Moreover, for any $u \in S_m$, there exist a_u, b_u such that $u \sqsubseteq a_u \cdot b_u$, $a_u \models \varphi$ and $b_u \models \psi$. Therefore, $m \models \varphi * \psi$.
- $\chi = \varphi \mathrel{-\!*} \psi$: for any n such that $n \models \varphi$, let $S_{n\cdot m} = \{n \cdot m' \mid m' \in S\}$. Then by the continuity axiom, $S_{n\cdot m} \in J(n \cdot m)$. For any $m' \in S$, since $m' \models \varphi \mathrel{-\!*} \psi$, $n \cdot m' \models \psi$. That is, for any $u \in S_{n\cdot m}$, $u \models \psi$. By the induction hypothesis, $n \cdot m \models \psi$.

□

The class of models in GTMs includes the models in elementary preordered commutative monoids, given in § 3.2, in the following sense:

Proposition 8 *For any preordered commutative monoid* $(M, \sqsubseteq, \cdot, e)$*, let* $J(m) = \{\, \{m'\} \mid m' = m \,\}$*. Then*

- $(M, \sqsubseteq, \cdot, e, J)$ *satisfies all of the axioms in this section;*
- *(K) implies (Sh); and*
- *when an interpretation makes (K) hold for atomic propositions, the interpretations of* $\varphi \vee \psi, \bot, \varphi * \psi, I$ *can be simplifed as follows:*

$$\begin{array}{rll} m \models \bot & \textit{iff} & \textit{never} \\ m \models \varphi \vee \psi & \textit{iff} & m \models \varphi \textit{ or } m \models \psi \\ m \models \varphi * \psi & \textit{iff} & \textit{there exists } n_\varphi, n_\psi \textit{ such that} \\ & & m \sqsubseteq n_\varphi \cdot n_\psi,\ n_\varphi \models \varphi \textit{ and } n_\psi \models \psi \\ m \models I & \textit{iff} & m \sqsubseteq e \end{array}$$

□

BI is sound and complete with respect to GTM models.

Proposition 9 (soundness) *For any* **BI** *propositions* φ *and* ψ*, if* $\varphi \vdash \psi$*, then* $\varphi \models \psi$ *in any GTM model.* □

Proof It is standard that $\top$, $\wedge$, $\to$, $\bot$, $\vee$ induce a Heyting algebra . We show that I, $*$ and $-\!\!*$ induce a residuated commutative monoid structure

- $(I, *)$ induce a monotone commutative monoidal structure

$$\frac{}{(\varphi * \psi) * \chi \models \varphi * (\psi * \chi)}\,1 \qquad \frac{}{\varphi * (\psi * \chi) \models (\varphi * \psi) * \chi}\,2$$

$$\frac{}{\varphi * I \models \varphi}\,3 \qquad \frac{}{\varphi \models \varphi * I}\,4$$

$$\frac{}{\varphi * \psi \models \psi * \varphi}\,5 \qquad \frac{\varphi \models \psi \quad \chi \models \rho}{\varphi * \chi \models \psi * \rho}\,6$$

1. the proof of this case follows from the following lemma:

   Lemma For any $m \in M$ and **BI** propositions φ_0, ψ_0 and χ_0, $m \models (\varphi_0 * \psi_0) * \chi_0$ iff there exists $S \in J(m)$ such that for any $m' \in S$, there exists $a_{m'}$, $b_{m'}$ and $c_{m'}$ in M such that $m' \sqsubseteq (a_{m'} \cdot b_{m'}) \cdot c_{m'}$, $a_{m'} \models \varphi_0$, $b_{m'} \models \psi_0$ and $c_{m'} \models \chi_0$.

   Suppose the above lemma holds. Then, by the associativity and commutativity of $\cdot$, $m \models (\varphi * \psi) * \chi$ iff $m \models (\psi * \chi) * \varphi$. As will be shown in 5, this is equivalent to $m \models \varphi * (\psi * \chi)$. The proof of the above lemma proceeds using axioms of a Grothendieck topology, as follows:

   if: for any $m' \in S$, since $\{a_{m'} \cdot b_{m'}\} \in J(a_{m'} \cdot b_{m'})$ by the maximality axiom, $a_{m'} \cdot b_{m'} \models \varphi_0 * \psi_0$. Therefore, $m \models (\varphi_0 * \psi_0) * \chi_0$.

   only if: since $m \models (\varphi_0 * \psi_0) * \chi_0$, there exists $S \in J(m)$ such that for any $m' \in S$, there exist $n_{m'}$ and $c_{m'}$ such that $m' \sqsubseteq n_{m'} \cdot c_{m'}$, $n_{m'} \models \varphi_0 * \psi_0$ and $c_{m'} \models \chi_0$. We'll show that for any m' in S, there exists $S_{m'}$ such that for any $d \in S_{m'}$, there exist $a_{m'}, b_{m'}$ satisfying that $d \sqsubseteq (a_{m'} \cdot b_{m'}) \cdot c_{m'}$, $a_{m'} \models \varphi$ and $b_{m'} \models \psi$. Then, the conclusion follows from $S_m = \bigcup_{m' \in S} S_{m'}$, which is in $J(m)$ by the transitivity axiom. Let's choose m' in S. Since $n_{m'} \models \varphi_0 * \psi_0$, there exists $S_{n_{m'}}$ such that for any $u \in S_{n_{m'}}$, there exist a_u and b_u satisfying that $u \sqsubseteq a_u \cdot b_u$, $a_u \models \varphi_0$ and $b_u \models \psi_0$. Let $S_{n_{m'} \cdot c_{m'}} = \{u \cdot c_{m'} \mid u \in S_{n_{m'}}\}$. Then, by the continuity of $\cdot$, $S_{n_{m'} \cdot c_{m'}} \in J(n_{m'} \cdot c_{m'})$. Since $m' \sqsubseteq n_{m'} \cdot c_{m'}$, by the stability axiom, there exists $S_{m'} \in J(m')$ such that for any $d \in S_{m'}$, $d \sqsubseteq u \cdot c_{m'}$ for some $u \in S_{n_{m'}}$, which, by the monotonicity of $\cdot$, implies that $d \sqsubseteq (a_u \cdot b_u) \cdot c_{m'}$.

2. this case is handled while proving the case 1.
3. for any $m \in M$ such that $m \models \varphi * I$, there exists $S \in J(m)$ such that for any $m' \in S$, there exist $a_{m'}, b_{m'}$ in M such that $m' \sqsubseteq a_{m'} \cdot b_{m'}$, $a_{m'} \models \varphi$ and $b_{m'} \models I$. By the interpretation of I, for any $m' \in S$, there exists $S_{b_{m'}} \in J(b_{m'})$ such that for any $u \in S_{b_{m'}}$, $u \sqsubseteq e$. By the continuity of $\cdot$, for any $m' \in S$, $\{a_{m'} \cdot u \mid u \in S_{b_{m'}}\} \in J(a_{m'} \cdot b_{m'})$. For any $m' \in S$ and $u \in S_{b_{m'}}$, since $a_{m'} \cdot u \sqsubseteq a_{m'} \cdot e = a_{m'}$ and $a_{m'} \models \varphi$, by (K), $a_{m'} \cdot u \models \varphi$. Therefore, by (Sh), $a_{m'} \cdot b_{m'} \models \varphi$, and since $m' \sqsubseteq a_{m'} \cdot b_{m'}$, (K) implies that $m' \models \varphi$ for all $m' \in S$.
4. for any $m \in M$ such that $m \models \varphi$, since $m = m \cdot e$, $\{m \cdot e\} \in J(m)$. Since $\{e\} \in J(e)$ and $e \sqsubseteq e$, $e \models I$. Therefore, $m \models \varphi * I$.
5. for any $m \in M$ such that $m \models \varphi * \psi$, there exists $S \in J(m)$ such that for any $m' \in S$, there exist $a_{m'}, b_{m'}$ in M such that $a_{m'} \models \varphi$, $b_{m'} \models \psi$ and $m' \sqsubseteq a_{m'} \cdot b_{m'}$. Since $\cdot$ is commutative, for any $m' \in S$, $m' \sqsubseteq b_{m'} \cdot a_{m'}$. Therefore, $m \models \varphi * \psi$.
6. for any $m \in M$ such that $m \models \varphi * \chi$, there is $S \in J(m)$ such that for any $m' \in S$, there exist $a_{m'}, c_{m'}$ in M such that $a_{m'} \models \varphi$, $c_{m'} \models \chi$ and $m' \sqsubseteq a_{m'} \cdot c_{m'}$. Since $\varphi \models \psi$ and $\chi \models \rho$, for any $m' \in S$, $a_{m'} \models \psi$ and $c_{m'} \models \rho$. Therefore, $m \models \psi * \rho$.

- $(*, -\!\!*)$ induce a residuated (closed) structure.

$$\frac{\varphi * \psi \models \chi}{\varphi \models \psi -\!\!* \chi}\,1 \qquad \frac{\varphi \models \psi -\!\!* \chi}{\varphi * \psi \models \chi}\,2$$

- for any $m, n \in M$ such that $m \models \varphi$ and $n \models \psi$, by the maximality axiom, $\{m \cdot n\}$ is in $J(m \cdot n)$, from which it follows that $m \cdot n \models \varphi * \psi$. Since $\varphi * \psi \models \chi$, $m \cdot n \models \chi$.
- for any $m \in M$ such that $m \models \varphi * \psi$, by the interpretation of $*$, there exists $S \in J(m)$ such that for any $m' \in S$, there exist $a_{m'}$ and $b_{m'}$ in S such that $m' \sqsubseteq a_{m'} \cdot b_{m'}$, $a_{m'} \models \varphi$ and $b_{m'} \models \psi$. Since $\varphi \models \psi \multimap \chi$, for any $m' \in S$, $a_{m'} \models \psi \multimap \chi$, from which it follows that $a_{m'} \cdot b_{m'} \models \chi$. By (K), $m' \models \chi$ for all $m' \in S$. By (Sh), $m \models \chi$.

□

Proposition 10 (completeness) *For any two* **BI** *propositions, if $\varphi \models \psi$ in all GTM models, then $\varphi \vdash \psi$.*

Proof The proof proceeds in a similar way to that for the completeness of $(\bot, \vee)$-free fragments, which can be seen, essentially, as constructing a complete model and using Yoneda embedding. Here, in contrast to the term model described for sheaves, disjunction is handled via the Grothendieck topology, J. The treatment of additives is standard, following the treatment for intuitionistic logic [20]. We present the completeness argument for intuitionistic as well as substructural connectives, in order to be self contained.

Define a GTM as follows:

- M is an equivalent class of a proposition φ, written $[\varphi]$, with respect to the relation given by provability;
- $[\varphi] \sqsubseteq [\psi]$ iff $\varphi \vdash \psi$. It can be easily shown that the choice of φ and ψ doesn't matter;
- $[\varphi] \cdot [\psi] = [\varphi * \psi]$. Also, it can be easily shown that the choice doesn't matter;
- $e = [I]$;
- $J([\varphi])$ is a collection of a finite (possibly empty) family $\{[\varphi_1], \ldots, [\varphi_n]\}$ such that $[\varphi_i] \sqsubseteq [\varphi]$ for all i and $[\varphi] \sqsubseteq [\varphi_1 \vee \ldots \vee \varphi_n]$. Here again, the choice doesn't matter.

We claim that the above entities do indeed satisfy all of the conditions required for a model. It is straightforward to show that $(M, \sqsubseteq, \cdot, e)$ is a preordered commutative monoid and that J satisfies the sieve and maximality axioms. We deal with the other three conditions.

- *Stability*: for any $[\varphi], [\psi] \in M$ and $\{[\varphi_l]\}_{l \in L} \in J([\varphi])$ such that $[\psi] \sqsubseteq [\varphi]$, let's consider the family $\{[\varphi_l \wedge \psi]\}_{l \in L}$. Since for any $l \in L$, $[\varphi_l \wedge \psi] \sqsubseteq [\psi]$ and $[\psi] \sqsubseteq [\bigvee_{l \in L}(\varphi_l \wedge \psi)]$, the family $\{[\varphi_l \wedge \psi]\}_{l \in L}$ belongs to $J([\psi])$. Moreover, $[\varphi_l \wedge \psi] \sqsubseteq [\varphi_l]$ for all $l \in L$, from which the other requirement for the stability axiom follows.
- *Transitivity*: for any $[\varphi] \in M$, $\{[\varphi_l]\}_{l \in L} \in J([\varphi])$ and $\{\{[\varphi_l^k]\}_{k \in K_l} \in J([\varphi_l])\}_{l \in L}$, let $S = \{[\varphi_l^k]\}_{l \in L, k \in K_l}$. From the definition of J, for any $l \in L$ and $k \in K_l$, $[\varphi_l^k] \sqsubseteq [\varphi_l] \sqsubseteq [\varphi]$. Again, from the definition of J, $[\varphi] \sqsubseteq [\bigvee_{l \in L} \varphi_l] \sqsubseteq [\bigvee_{l \in L} \bigvee_{k \in K_l} \varphi_l^k]$, which implies $[\varphi] \sqsubseteq [\bigvee_{l \in L, k \in K_l} \varphi_l^k]$. Therefore, S is in $J([\varphi])$.
- *Continuity*: for any $[\varphi], [\psi] \in M$ and $\{[\varphi_l]\}_{l \in L} \in J([\varphi])$, let's consider the family $\{[\varphi_l * \psi]\}_{l \in L}$. Then $[\varphi_l * \psi] \sqsubseteq [\varphi * \psi]$ for any $l \in L$ and $[\varphi * \psi] \sqsubseteq [(\bigvee_{l \in L} \varphi_l) * \psi] = [\bigvee_{l \in L}(\varphi_l * \psi)]$.

Let the interpretation $[\![-]\!]$ of atomic propositions be given by $[\![\mathrm{p}]\!] = \{[\varphi] \mid \varphi \vdash \mathrm{p}\}$. Notice that $[\![-]\!]$ satisfies (K) and (Sh). The resulting model has the following property:

For any two propositions φ_0 and ψ_0, $[\varphi_0] \models \psi_0$ iff $\varphi_0 \vdash \psi_0$.

Before considering why the above property holds, notice that the completeness result follows from it in the usual way. We show the above property by the induction on the structure of ψ_0.

- $\psi_0 = \mathrm{p}$: this case follows from the definition of $[\![-]\!]$.
- $\psi_0 = \top$: both $[\varphi_0] \models \top$ and $\varphi_0 \vdash \top$ always hold.

- $\psi_0 = \varphi \wedge \psi$:
  $[\varphi_0] \models \varphi \wedge \psi$ iff $[\varphi_0] \models \varphi$ and $[\varphi_0] \models \psi$ iff (by the induction hypothesis) $\varphi_0 \vdash \varphi$ and $\varphi_0 \vdash \psi$ iff $\varphi_0 \vdash \varphi \wedge \psi$.

- $\psi_0 = \varphi \to \psi$:

  if: for any $[\varphi_1]$ such that $[\varphi_1] \sqsubseteq [\varphi_0]$ and $[\varphi_1] \models \varphi$, $\varphi_1 \vdash \varphi$ by the induction hypothesis, From the definition of $\sqsubseteq$, $\varphi_1 \vdash \varphi_0$. Therefore, $\varphi_1 \vdash \varphi \to \psi$ and $\varphi_1 \vdash \psi$. Again, by the induction hypothesis, $[\varphi_1] \models \psi$;

  only if: since $\varphi_0 \wedge \varphi \vdash \varphi$, $[\varphi_0 \wedge \varphi] \models \varphi$ by the induction hypothesis. Since $[\varphi_0 \wedge \varphi] \sqsubseteq [\varphi_0]$, $[\varphi_0 \wedge \varphi] \models \psi$. Again, by the induction hypothesis, $\varphi_0 \wedge \varphi \vdash \psi$. Therefore, $\varphi_0 \vdash \varphi \to \psi$.

- $\psi_0 = \varphi \vee \psi$:

  if: consider $S = \{[\varphi_0 \wedge \varphi], [\varphi_0 \wedge \psi]\}$. Then, $[\varphi_0 \wedge \varphi] \sqsubseteq [\varphi_0]$ and $[\varphi_0 \wedge \psi] \sqsubseteq [\varphi_0]$ and $[\varphi_0] \sqsubseteq [\varphi_0 \wedge (\varphi \vee \psi)] = [(\varphi_0 \wedge \varphi) \vee (\varphi_0 \wedge \psi)]$. Therefore, $S \in J([\varphi_0])$. Moreover, by the induction hypothesis, $[\varphi_0 \wedge \varphi] \models \varphi$ and $[\varphi_0 \wedge \psi] \models \psi$. Thus, $[\varphi_0] \models \varphi \vee \psi$;

  only if: since $[\varphi_0] \models \varphi \vee \psi$, there exist $S \in J([\varphi_0])$ such that for any $[\varphi'] \in S$, $[\varphi'] \models \varphi$ or $[\varphi'] \models \psi$. By induction hypothesis, for any $\varphi' \in S$, $\varphi' \vdash \varphi$ or $\varphi' \vdash \psi$, which implies $\varphi' \vdash \varphi \vee \psi$. $\bigvee_{\varphi' \in S} \varphi' \vdash \varphi \vee \psi$ follows from this. Since $[\varphi_0] \sqsubseteq [\bigvee_{\varphi' \in S} \varphi']$, $\varphi_0 \vdash \varphi \vee \psi$.

- $\psi_0 = \perp$: $[\varphi_0] \models \perp$ iff $\emptyset \in J([\varphi_0])$ iff $[\varphi_0] \sqsubseteq [\perp]$ iff $\varphi_0 \vdash \perp$. This case is the counterpart to the $\emptyset \models \perp$ case in the sheaf-theoretic semantics discussed in § 6.2.

- $\psi_0 = I$:

  if: $\{[\varphi_0]\} \in J([\varphi_0])$ and $[\varphi_0] \sqsubseteq e = [I]$ because $\varphi_0 \vdash I$. Therefore, $[\varphi_0] \models I$;

  only if: since $[\varphi_0] \models I$, there exists $\{[\varphi_l]\}_{l \in L} \in J([\varphi_0])$ such that $[\varphi_l] \sqsubseteq e = [I]$ for any $l \in L$, which implies $\bigvee_{l \in L} \varphi_l \vdash I$. Since $[\varphi_0] \sqsubseteq [\bigvee_{l \in L} \varphi_l]$, $\varphi_0 \vdash \bigvee_{l \in L} \varphi_l$. Therefore, $\varphi_0 \vdash I$.

- $\psi_0 = \varphi * \psi$:

  if: $\{[\varphi_0]\} \in J([\varphi_0])$ and $[\varphi_0] \sqsubseteq [\varphi] \cdot [\psi]$. Moreover, by the induction hypothesis, $[\varphi] \models \varphi$ and $[\psi] \models \psi$. Therefore, $[\varphi_0] \models \varphi * \psi$;

  only if: since $[\varphi_0] \models \varphi * \psi$, there exists $\{[\varphi_l]\}_{l \in L} \in J([\varphi_0])$ such that for any $l \in L$, there exist $[\sigma_l], [\tau_l]$ such that $[\varphi_l] \sqsubseteq [\sigma_l] \cdot [\tau_l]$, $[\sigma_l] \models \varphi$ and $[\tau_l] \models \psi$. By the induction hypothesis, $\sigma_l \vdash \varphi$ and $\tau_l \vdash \psi$ for any $l \in L$. For any $l \in L$, since $[\varphi_l] \sqsubseteq [\sigma_l * \tau_l]$, $\varphi_l \vdash \varphi * \psi$. Since $\varphi_0 \vdash \bigvee_{l \in L} \varphi_l$, $\varphi_0 \vdash \varphi * \psi$.

- $\psi_0 = \varphi \multimap \psi$:

  if: for any $[\varphi_1]$ such that $[\varphi_1] \models \varphi$, by induction hypothesis, $\varphi_1 \vdash \varphi$. Therefore, $\varphi_0 * \varphi_1 \vdash \psi$. Again, by the induction hypothesis, $[\varphi_0 * \varphi_1] \models \psi$. Equivalently, $[\varphi_0] \cdot [\varphi_1] \models \psi$;

  only if: by the induction hypothesis, $[\varphi] \models \varphi$. Since $[\varphi_0] \models \varphi \multimap \psi$, $[\varphi_0] \cdot [\varphi] \models \psi$. By the induction hypothesis again, $\varphi_0 * \varphi \vdash \psi$. Therefore, $\varphi_0 \vdash \varphi \multimap \psi$.

$\square$

We conclude this part with a simple example, a specific counter-model to the entailment,

$$((\mathrm{p} \multimap \perp) \to \perp) \wedge ((\mathrm{q} \multimap \perp) \to \perp) \models (\mathrm{p} * \mathrm{q} \multimap \perp) \to \perp,$$

used in Proposition 2. We define a preordered monoid $\mathcal{M} = (M, \cdot, \sqsubseteq)$, where

- the carrier set $M = \{ e, a, \perp \}$;
- the order is $e \sqsupseteq \perp \sqsubseteq a$;

- the multiplication is

$\cdot$	e	a	$\perp$
e	e	a	$\perp$
a	a	$\perp$	$\perp$
$\perp$	$\perp$	$\perp$	$\perp$

- the Grothendieck topology is

$$J(\perp) = \{\{\perp\}, \emptyset\}, J(e) = \{\{e\}\}, J(a) = \{\{a\}\}$$

Define an interpretation and forcing relation as follows:

- $m \models$ p iff $m = a$ or $m = \perp$;
- $m \models$ q iff $m = a$ or $m = \perp$.
- $m \models \perp$ iff $m = \perp$.

Now, $e \models (\mathrm{p} \twoheadrightarrow \perp) \to \perp$ iff for all $n \sqsubseteq e$ such that $e \neq \perp$ there is an l such that $l \models$ p and $n \cdot l \neq \perp$ iff there exists l such that $l \models$ p and $l \neq \perp$. Since a is such an l, we have $e \models (\mathrm{p} \twoheadrightarrow \perp) \to \perp$. However, $e \models (\mathrm{p} * \mathrm{q} \twoheadrightarrow \perp) \to \perp$ iff for any $n \sqsubseteq e$ such that $n \neq \perp$ there is an l such that $l \models \mathrm{p} * \mathrm{q}$ and $l \cdot n \neq \perp$ iff there are l, l' such that $l \models$ p, $l' \models$ q and $l \cdot l' \neq \perp$ which cannot be so because, for any l and l', if $l \models$ p and $l' \models$ q, then $l \cdot l' = \perp$. Therefore, $e \not\models (\mathrm{p} * \mathrm{q} \twoheadrightarrow \perp) \to \perp$

Therefore, $e \models ((\mathrm{p} \twoheadrightarrow \perp) \to \perp) \wedge ((\mathrm{q} \twoheadrightarrow \perp) \to \perp)$ but $e \not\models ((\mathrm{p} * \mathrm{q} \twoheadrightarrow \perp) \to \perp)$ in this model.

6.4 The Pointer Model as a Grothendieck Sheaf

In § 5, we presented a model where the combining operation $\cdot$ is a partial function. We show that this can be understood as a Grothendieck sheaf model.

First, let $H_\perp$ be the set of heaps, extended with a new least element, $\perp$. We can define an operation $\cdot$ in which $h \cdot h'$ is the union of $h, h' \in H$ if they are disjoint and $\perp$ otherwise. Also, $\cdot$ is strict in both arguments and the unit is again the empty heap. The ordering we take is the flat one, in which $\perp$ is least and all other elements are incomparable.

We can define a Grothendieck simple topology on $H_\perp$, by setting

$$\begin{array}{lcll} J(\perp) & = & \{\{\perp\}, \emptyset\} & \\ J(m) & = & \{\{m\}\} & \text{if } m \neq \perp \end{array}$$

The points-to relation is extended so that $\perp$ always forces it. Notice that since $J(\perp)$ contains $\emptyset$, it follows from the semantic clauses that $\perp \models \varphi$ always holds.

The connection between the pointer model and this sheaf presentation can then be stated as follows:

For every $h \in H$, $h \models \varphi$ in the sheaf model just given iff $h \models \varphi$ in the pointer model.

This does not mention $\perp$ but, because of the way it is treated in the topology, the two models do indeed agree on logical consequence:

$\psi \models \varphi$ in the sheaf model just given iff $\psi \models \varphi$ in the pointer model.

Finally, the pointer model of Reynolds [32] can also be seen as a sheaf model. The underlying set of worlds is $H_\perp$ as above, but this time the ordering on worlds is the one where $h \sqsubseteq h'$ if the graph of h is a superset of the graph of h'. This is an intuitionistic model, whereas the previous one provides a model of Boolean **BI**.

7 On Resource and Substructural Logic

In the introduction we acknowledged the influence of previous work on relevant and linear logic. We conclude by discussing the relation of our work to theirs in more detail and by providing a retrospective on resources and substructural logics.

Some of the ideas of resource semantics can certainly be seen in the relevantists' work [7]. But they do not provide much discussion of the sense of "resource" or give convincing example models in which that sense is clear. Relevantists would likely be well aware of this point; indeed, they often make light of informal interpretations attached to their semantics (an important exception being [25]). We only point out that if one starts from resource, and considers concrete examples, then it is very difficult to maintain the rule of Contraction. Our discussion of areas of memory at the beginning of the paper provides a clear example.

Of course, relevantists and others have considered logics without Contraction from time to time, and a decisive step forward was made in Girard's imaginative description of the impact of Contraction in terms of action [11]. Our rationale for controlling Contraction is reminiscent of Girard's and was certainly influenced by it. However, if one looks beyond surface similarity there are critical differences. Principal among these is our emphasis on truth-based semantics. Now linear logic does have a semantics, the phase semantics, which is formally similar to ours in its treatment of multiplicatives. However, Girard explicitly de-emphasizes the phase semantics, and few interesting examples of it have been given. Perhaps this is because of its treatment of additives, which denies distribution and, more significantly, $\rightarrow$. (We stress that this point is neither at odds with linear logic's analysis of classical proof theory nor with the many interesting models of linear logic's proof theory which clearly establish the importance of its corresponding λ-calculi.)

Although constructive logic is very important, we claim that the significance of controlling Contracton comes out directly, and indeed quickly, from the truth-based point of view advanced here: "this area of memory contains three cells" makes immediate computational sense, without any appeal to proof theory or constructivism. This frees us to adopt a particularly simple-minded interpretation of the surrounding connectives, which amounts to a semantics of classical or intuitionistic logic. Too often, the possibilities offered by restricting Contraction have been short-circuited by an over-reliance on constructive methods, in situations where they are scarcely relevant.

Aside from the truth-versus-proof issue, there is the question of what the "resources" actually are. Linear logicians have often appealed to resource but, again, the sense of the word is usually not given. So it is worth considering the question: what kinds of resources do linear logic proofs manipulate ? It is difficult to say with confidence — and a careful anaysis would be welcome — but there appears to be a good analogy with what in operating systems are called *temporary resources*, which are *produced and consumed* by processes. (The terminology "consumable resources" is sometimes also used.) Temporary resources include messages, the contents of files, or the state of the computer store. Consumption is certainly evident in several models of linear logic, including the original coherence space model [10] (where the temporary resources are tokens) and the concurrency reading [36] (where the temporary resources are messages).

In the Petri net model of **BI**, we have seen that consumption can be accounted for in a broadly similar way, by internalizing state transitions using an intuitionistic ordering. We also argued that this is unnatural. There are highly successful logics that (in the present terminology) deal very well with consumption of temporary resources. These include Hoare logic and temporal and modal logics of processes. It is difficult to see how a substructural logic could, by itself, improve on these accounts; at least, no demonstration has yet been given.

This criticism applies equally to linear logic and to **BI**. So, if substructural logics give an impoverished method of talking about resource consumption, where does that leave us ? (Again we emphasize that we are speaking of logic here, not type theory, where the consumptive aspect has played a useful role in the discovery of interesting models [37, 17] or type theories [38, 16].)

This brings us to *permanent* or *long-lived* resources. Examples include files, storage and external devices, where sharing rather than consumption is the operative concept. Notice how temporary and permanent resources are sometimes linked: the state of the store is something that is consumed by assignment, while the store itself is shared. This distinction between permanent and temporary resources is perhaps not hard-and-fast; for example, a file can itself be created and destroyed, so it has a temporary aspect. Nonetheless, as an organizing principle, it is a useful distinction.

Based on this discussion, we offer a general suggestion on substructural logic and resource: substructural logic can be used to effectively talk about access to resources, including long-lived resources, where control of structural rules enables specifications to remain local. By this we mean that the meaning of a formula can be judged with respect to resources in a confined area and then combined with formulæ that talk about other resources. However, for describing consumption other approaches seem better suited: *i.e.*, substructural logic should be used to supplement standard approaches to specification rather than to supplant them.

In conclusion, a degree of modesty is called for in considering the accomplishments and prospects of "resource-conscious logics" or "resource semantics". Early in the paper, we remarked that resource is a multi-faceted notion: There are a variety of important concepts, including sharing, ownership, competition, co-operation, authorization and location, most of which we have not touched on at all. For example, although sharing is exhibited clearly in a specific model, the pointer model, it is not part of our axiomatization. So, although we believe that we have demonstrated that our approach goes well beyond previous logics in some areas, what we have done is but a start. We hope that it will prove a sound basis for further enquiry.

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