Category Theory	**Lars Birkedal**
Fall 2000	**Thomas Hildebrandt**

Final Take-Home Exam

Timing: The exam **begins** on **Thursday January 18, 2001, 10:00 AM**. Between 10:00 AM and 11:00 AM, you can pick up the exam from Thomas Hildebrandt's office (Room 2.26 at the IT University, Glentevej 67, DK-2400 Copenhagen NV). At 10:00 AM, the exam will also be posted on the course home page
`www.it-c.dk/~birkedal/teaching/category-theory-Fall-2000/`
(under "assignments").
You must **hand-in** your answers before **Friday January 19, 2001, 10:00 AM** in the "Studieadministration."
That is, you have 24 hours to work on the exam. (The exam, however, has been designed so that an average student should be able to complete it in 6 hours.)

Pensum: Everything in the Course Notes (Jaap van Oosten: *Category Theory in Computer Science*), except Section 7.3.

Rules: You are allowed to work on the exam anywhere you want, make use of all course material (available on the course home page), and whatever other material you may find helpful. You are *not* allowed to collaborate or get help from anyone. Remember to write your name on your answer sheets.

A partial equivalence relation $\approx$ on a set A is a binary relation on A that is symmetric ($a \approx a' \implies a' \approx a$) and transitive ($a \approx a' \wedge a' \approx a'' \implies a \approx a''$).

Let $\mathbb{C}$ be a cartesian closed category. We define the category $\mathbf{PER}(\mathbb{C})$ of *partial equivalence relations over* $\mathbb{C}$ as follows. Objects are pairs $(X, \approx_X)$ with $X \in \mathbb{C}$ an object of $\mathbb{C}$ and $\approx_X$ a partial equivalence relation on the set $\mathbb{C}(1, X)$ (here 1 denotes the terminal object in $\mathbb{C}$). Morphisms $f\colon (X, \approx_X) \to (Y, \approx_Y)$ are equivalence classes $f = [F]$ of morphisms $F\colon X \to Y$ in $\mathbb{C}$ such that

$$\forall x, x' \in \mathbb{C}(1, X).\ x \approx_X x' \implies F \circ x \approx_Y F \circ x',$$

with two such F and F' equivalent iff

$$\forall x \in \mathbb{C}(1, X).\ x \approx_X x \implies F \circ x \approx_Y F' \circ x.$$

Question 1 Prove that $\mathbf{PER}(\mathbb{C})$ is a category (what are the identities and what is composition ?).

Now we will give an equivalent defintion of $\mathbf{PER}(\mathbb{C})$: Define the category $\mathbf{Mod}(\mathbb{C})$ of *modest sets over* $\mathbb{C}$ to have objects pairs (A, X, E_A) with $A \in \mathbf{Set}$ an ordinary set, $X \in \mathbb{C}$ an object of $\mathbb{C}$, and $E_A \colon A \to P(\mathbb{C}(1, X))$ a funtion from A into the powerset of $\mathbb{C}(1, X)$ satisfying that

1. $\forall a, a' \in A.\ E_A(a) \cap E_A(a') \neq \emptyset \implies a = a'$;
2. $\forall a \in A.\ E_A(a) \neq \emptyset$.

For an object (A, X, E_A), we refer to A as the *underlying set* and to E_A as the *existence predicate*. Morphism from (A, X, E_A) to (B, Y, E_B) are functions $f \colon A \to B$ in $\mathbf{Set}$ for which there exists a morphism $g \colon X \to Y$ in $\mathbb{C}$ which *tracks* f in the sense that:

$$\forall a \in A.\ \forall x \in E_A(a).\ g \circ x \in E_B(f(a)).$$

We say that g is a *realizer* for f.

Question 2 Prove that $\mathbf{Mod}(\mathbb{C})$ is a category (what are the identities and what is composition ?).

Question 3 Prove that $\mathbf{Mod}(\mathbb{C})$ is equivalent to $\mathbf{PER}(\mathbb{C})$ (written $\mathbf{Mod}(\mathbb{C}) \simeq \mathbf{PER}(\mathbb{C})$).

Question 4 Prove that $\mathbf{Mod}(\mathbb{C})$ is cartesian closed. (Since $\mathbf{Mod}(\mathbb{C}) \simeq \mathbf{PER}(\mathbb{C})$ you can show cartesian closure either for the category of modest sets or for the category of partial equivalence relations, as you prefer.) **Hint:** given two modest sets $M = (A, X, E_A)$ and $N = (B, Y, E_B)$, the exponential N^M is (C, Y^X, E_C) with

$$\begin{aligned} C &= \{\, f \in B^A \mid \exists g \in \mathbb{C}(1, Y^X).\ \forall a \in A.\ \forall x \in E_A(a).\ \epsilon_{Y,X} \circ \langle g, x \rangle \in E_B(b) \,\} \\ E_C(f) &= \{\, g \in \mathbb{C}(1, Y^X) \mid \forall a \in A.\ \forall x \in E_A(a).\ \epsilon_{Y,X} \circ \langle g, x \rangle \in E_B(b) \,\}. \end{aligned}$$

Here $\epsilon_{Y,X}$ is evaluation map associated with the exponential object Y^X in $\mathbb{C}$.

In words, C is the set of functions from A to B for which there exists a realizer g which tracks f, and $E_C(f)$ is the set of realizers for f.

You may use the internal language $\mathcal{L}(\mathbb{C})$ of $\mathbb{C}$ when conducting the proof. Recall that, in the internal language, an $x \in \mathbb{C}(1, X)$ is an element of sort X and that $\epsilon_{Y,X} \circ \langle g, x \rangle$ is written in the internal language as $g(x)$.

Enjoy!
