Local Realizability Toposes and a Modal Logic for Computability

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Overview and Motivation

- An approach to computability on non-computable data
- $A_\sharp \subseteq A$
- Examples: Graph Model: $RE \subseteq P\mathbb{N}$, TRF $\subseteq \mathbb{N}^{\mathbb{N}}$, term models
- Setup:

$$\begin{tikzcd} & RT(A, A_\sharp) \arrow[dl, shift left] \arrow[dr] & \\ RT(A_\sharp) \arrow[ur, shift left=2] \arrow[ur, shift right=2] & & RT(A) \end{tikzcd}$$

Outline

- Realizability toposes for computability
- Geometry of realizability toposes for computability
- Axioms for localic local maps
- Modal logic for local maps
- Interpretation in realizability models

Definition of $RT(A, A_\sharp)$

1. Tripos $P\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$
   - non-standard predicates
     $$\varphi, \psi\colon I \to PA$$
     and Heyting pre-algebra operations as for tripos over A
   - $\varphi \vdash_I \psi$ iff
     $$\exists a \in A_\sharp . \forall i \in I . a\colon \varphi(i) \supset \psi(i)$$
2. $RT(A, A_\sharp) = \mathbf{Set}[P]$
   - Objects: $(X, \approx_X)$, $\approx_X\colon X \times X \to PA$, realizers for symmetry and transitivity in $A_\sharp$
   - Morphisms: $f\colon (X, \approx_X) \to (Y, \approx_Y)$ represented by functional relations $F\colon X \times Y \to PA$, realizers for functionality in $A_\sharp$
   - Slogan: “continuous objects, computable morphisms”

Geometry of realizability toposes

- Let $Q\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$ be the standard real. tripos on $A_\sharp$
- $\mathbf{Set}$-indexed functors between Q and P:

$$\Delta\colon Q \to P \quad \text{and} \quad \Gamma\colon P \to Q \quad \text{and} \quad \nabla\colon Q \to P$$

- Over 1:
  - $\Delta(\psi \in P\,A_\sharp) = \psi$
  - $\Gamma(\varphi \in P\,A) = A_\sharp \cap \varphi$
  - $\nabla(\psi \in P\,A_\sharp) = \bigcup_{\varphi \in P\,A} \big(\varphi \wedge (A_\sharp \cap \varphi \supset \psi)\big)$
    (where $\wedge$ and $\supset$ are calculated as in $P(1)$)
- Theorem:
  - (Δ, Γ) is a geometric morphism of triposes from P to Q.
  - (Γ, ∇) is a geometric morphism of triposes from Q to P.
  - For all $I \in \mathbf{Set}$, Δ_I and ∇_I are both full and faithful.

At the topos level

$$\begin{tikzcd} RT(A_\sharp) & RT(A, A_\sharp) \arrow[l, "\Gamma"'] \arrow[from=1-1, to=1-2, bend left, "\Delta"] \arrow[from=1-1, to=1-2, bend right, "\nabla"'] \end{tikzcd}, \qquad \Delta \dashv \Gamma \dashv \nabla.$$

- Δ lex.
- Δ and ∇ full and faithful. $\Gamma \circ \nabla \cong 1 \cong \Gamma \circ \Delta$.
- So $(\Delta, \Gamma)\colon\ RT(A, A_\sharp) \to RT(A_\sharp)$ is connected surjective g.m.
- So $(\Gamma, \nabla)\colon\ RT(A_\sharp) \to RT(A, A_\sharp)$ is an embedding.

$RT(A_\sharp)$ equivalent to category of sheaves

$$
\begin{tikzcd}
 & RT(A, A_\sharp) \arrow[dl, "\Gamma" description] \arrow[dr, "\mathbf{a}" description] & \\
RT(A_\sharp) \arrow[ur, bend left, "\Delta"] \arrow[ur, bend right, "\nabla"'] \arrow[rr, bend left=10, "\mathbf{a}\,\nabla"] & & \mathrm{Sh}_j RT(A, A_\sharp) \arrow[ul, bend right, "i"'] \arrow[ul, bend left, "\Delta\Gamma i"] \arrow[ll, bend left=10, "\Gamma i"]
\end{tikzcd}
$$

$$\Delta \dashv \Gamma \dashv \nabla,$$

$$\Gamma i\,\mathbf{a} \dashv \mathbf{a} \dashv i,$$

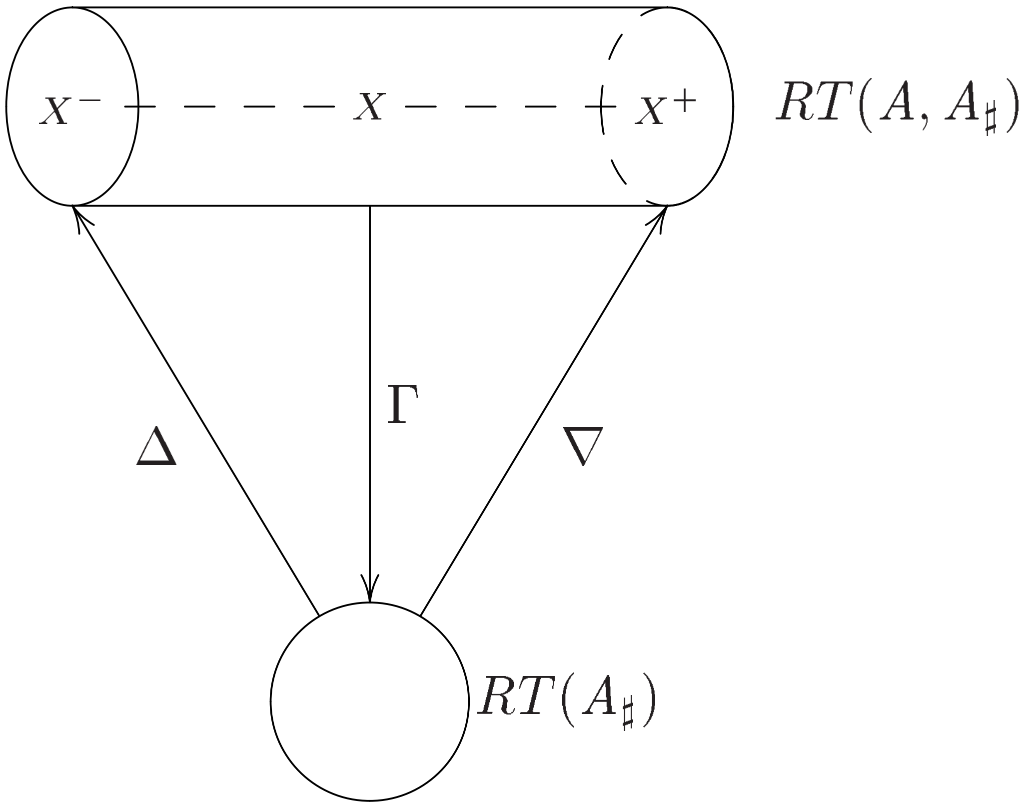
$$\mathbf{a}\,\nabla\Gamma i \cong id,$$

$$\Gamma i\,\mathbf{a}\,\nabla \cong id,$$

Δ f+f and lex,

∇ f+f.

“Adjoint Cylinder” (Lawvere)



Localic Toposes from Triposes

- Theorem: If $f = (f^*, f_*)\colon P \to Q$ is a geo. mor. of $\mathbb{C}$-triposes, then $\mathbb{C}[P]$ is localic over $\mathbb{C}[Q]$.

  Thus, for all $X \in \mathbb{C}[P]$, there exists $Y \in \mathbb{C}[Q]$ and a diagram

$$\begin{tikzcd} S \arrow[r, tail] \arrow[d, two heads] & f^*Y \\ X & \end{tikzcd}$$

  in $\mathbb{C}[P]$ presenting X as a subquotient of f^*Y.

- Thus, $RT(A, A_\sharp)$ is localic over $RT(A_\sharp)$ via (Δ, Γ).

- Conclusion: $(\Delta, \Gamma) : RT(A, A_\sharp) \to RT(A_\sharp)$ is a **localic local map of toposes**.

Axioms for Localic Local Maps, I

Given topos $\mathcal{E}$ with topology j.

Definitions:

- j is **principal** if, for all $X \in \mathcal{E}$, the closure operation on $Sub(X)$ has a left adjoint, called **interior**:

$$\mathring{U} \leq V \iff U \leq \overline{V} \qquad \text{in } Sub(X).$$

- $X \in \mathcal{E}$ is **open** iff $\mathring{X} \cong X$.
- $C \in \mathcal{E}$ is **discrete** if it is **coorthogonal** to all morphisms inverted by **a**:

$$\begin{tikzcd} & X \arrow[d, "e"] \\ C \arrow[ur, dashed, "f'"] \arrow[r, "f"'] & Y \end{tikzcd} \qquad \mathbf{a}(e) \text{ iso}$$

Axioms for Localic Local Maps, II

Axiom 1. The topology j is principal.

Axiom 2. For all $X \in \mathcal{E}$, there exists a discrete object C and a diagram

$$\begin{tikzcd} S \arrow[r, tail] \arrow[d, two heads] & C \\ X & \end{tikzcd}$$

in $\mathcal{E}$, presenting X as a subquotient of C.

Axiom 3. For all discrete $C \in \mathcal{E}$, if $X \rightarrowtail C$ is open, then X is also discrete.

Axiom 4. For all discrete $C, C' \in \mathcal{E}$, the product $C \times C'$ is again discrete.

Soundness and Completeness

Theorem:

Let $\mathcal{E}$ be a topos with a topology j satisfying the Axioms 1–4 for localic local maps.

The category of discrete objects $\mathrm{D}_j\mathcal{E}$ is coreflective in $\mathcal{E}$, that is, the inclusion $\Delta\colon \mathrm{D}_j\mathcal{E} \to \mathcal{E}$ has a right adjoint. Moreover, $\mathrm{D}_j\mathcal{E}$ is a topos, Δ is left exact, and $(\Delta, \Gamma)\colon \mathcal{E} \to \mathrm{D}_j\mathcal{E}$ is a localic local map.

Theorem:

Every localic local map of toposes satisfies Axioms 1-4 for localic local maps.

Sketch of soundness proof

The associated discrete object of an object $X \in \mathcal{E}$ is found as follows.

$$
\begin{tikzcd}
\mathring{K}_{\mathring{e}} \arrow[r, tail] & K_{\mathring{e}} \arrow[d, shift left] \arrow[d, shift right] & & \\
& \mathring{S} \arrow[r, tail, "m"] \arrow[dl] \arrow[d, two heads, "\mathring{e}"] & S \arrow[r, tail] \arrow[d, two heads, "e"] & C \\
\widetilde{\mathring{X}} \arrow[r, two heads, "h"'] & \mathring{X} \arrow[r, tail] & X &
\end{tikzcd}
$$

A modal logic for local maps, I

Given topos $\mathcal{E}$ with topology j satisfying axioms for localic local maps.

- Goal: Logic to reason about the two toposes $\mathcal{E}$ and $\mathrm{D}_j\mathcal{E}$.
- Problem: interior does not commute with pullback.
- A solution: consider

$$\begin{tikzcd} Pred \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Sub(\mathcal{E}) \arrow[d] \\ \mathrm{D}_j\mathcal{E} \arrow[r, "\Delta"'] & \mathcal{E}. \end{tikzcd}$$

  - types and terms: discrete objects and morphisms between such
  - predicates on type X are all predicates in $\mathcal{E}$ on ΔX

A modal logic for local maps, II

$$Sub_{\mathcal{E}}(\Delta X) \cong \mathcal{E}(\Delta X, \Omega_{\mathcal{E}}) \cong \mathrm{D}_j\mathcal{E}(X, \Gamma\Omega_{\mathcal{E}})$$

Hence:

- the resulting logic is tripos logic on the internal locale $\Gamma\Omega_{\mathcal{E}}$ in $\mathrm{D}_j\mathcal{E}$.
- the tripos is $H = \mathrm{D}_j\mathcal{E}(\_, \Gamma\Omega_{\mathcal{E}})$ and $\mathcal{E} = \mathrm{D}_j\mathcal{E}[H]$.

Interior commutes with pullback in $\begin{array}{c} Pred \\ \downarrow \\ \mathrm{D}_j\mathcal{E} \end{array}$, so gives a logical operation on predicates; we denote it by $\sharp$.

The topology j in $\mathcal{E}$ induces a closure operator, $\flat$, on predicates in this internal logic.

A modal logic for local maps, III

$$\frac{}{\Gamma \mid \sharp\varphi \vdash \varphi} \qquad \frac{}{\Gamma \mid \sharp\varphi \vdash \sharp\sharp\varphi}$$

$$\frac{}{\Gamma \mid \emptyset \vdash \sharp(T)} \qquad \frac{}{\Gamma \mid \sharp\varphi \wedge \sharp\psi \vdash \sharp(\varphi \wedge \psi)}$$

$$\frac{\Gamma \mid \sharp\varphi \vdash \psi}{\Gamma \mid \varphi \vdash \flat\psi}$$

From this one prove that $\sharp$ has the formal properties of box in S4. We therefore refer to this logic as a **modal logic for computability**. Further:

$$\frac{}{\Gamma \mid \sharp\exists x{:}X.\,\varphi \dashv\vdash \exists x{:}X.\,\sharp\varphi}$$

Internal Locale Viewpoint

Internally in $\mathrm{D}_j\mathcal{E}$:

$$\begin{tikzcd} \Gamma\Omega_{\mathcal{E}} \arrow[d, "\Gamma"] \\ \Omega_{\mathrm{D}_j\mathcal{E}} \arrow[u, bend left=60, "\Delta"] \arrow[u, bend right=60, "\nabla"'] \end{tikzcd} \qquad \Delta \dashv \Gamma \dashv \nabla \qquad \Gamma\Delta = 1 = \Gamma\nabla$$

- $\sharp = \Delta\Gamma$ — has S4 properties automatically
- the further adjoint means that $\sharp$ commutes with existentials

Interpretation in $RT(A_\sharp)$ and $RT(A, A_\sharp)$

- types and terms: objects and morphisms of $RT(A_\sharp)$
- predicates φ on type $(X, \approx)$ are $\varphi\colon X \to P\,A$, strict and extensional via realizers from $A_\sharp$
- $\sharp\varphi(x) = \varphi(x) \cap A_\sharp$
- $\flat\varphi(x) = \bigcup_{\psi \in P\,A}(\psi \wedge (\psi \cap A_\sharp \supset \varphi(x) \cap A_\sharp))$