# A General Notion of Realizability

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# Introduction

**Background:**

① Wanted to understand fundamental logical and type-theoretical properties of the category $\mathbf{Equ}$ of equilogical spaces (BBS98).

② Saw connection to standard Kleene-style realizability.

③ A general notion of realizability encompassing

- standard Kleene-style realizability
- "typed" realizability used for $\mathbf{Equ}$.

**More Motivation** in work of Longley (L99): comparisons of different computational models.

# Equilogical Spaces

Category $\mathbf{Equ}$ with

**objects** topological $T_0$-spaces with arbitrary equivalence relations

**morphisms** equivalence classes of continuous functions preserving the equivalence relations

Motivation:

- contains many categories of domains as used in denotational semantics
- contains many ordinary mathematical spaces, so allows study of computable analysis

# $\mathbf{Equ} \simeq \mathbf{Mod}(\mathbf{ALat})$

$\mathbf{Mod}(\mathbf{ALat})$:

**objects** $(X, A, E_X)$ with $X \in \mathbf{Set}$, $A \in \mathbf{ALat}$, and $E_X : X \to PA$ such that $E_X(x) \cap E_X(x') \neq \emptyset \implies x = x'$.

**morphisms** $(X, A, E_X) \to (Y, B, E_Y)$ are functions $f : X \to Y$ in $\mathbf{Set}$ such that there exists a $g : A \to B$ in $\mathbf{ALat}$ which realizes $f$:

$$\forall x \in X.\ \forall a \in E_X(x).\ g(a) \in E_Y(fx)$$

# General Notion of Realizability, I

Based on $\mathrm{WCPC}$-category — think of as

- objects are sets of realizers
- morphisms are realizers: a kind of possibly partial functions
- can form the product of sets of realizers
- for any two sets of realizers $X$ and $Y$ there is a set of realizers $[X \rightharpoonup Y]$ which contains the set of realizers from $X$ to $Y$

Any CCC (e.g. $\mathbf{ALat}$) is a $\mathrm{WCPC}$-category.

A $\mathrm{PCA}$ gives rise to a $\mathrm{WCPC}$-category.

# General Notion of Realizability, II

Un(i)typed (Impredicative) Realizability

① PCA $A$

② real. tripos $\begin{array}{c}\mathbf{UFam}(A)\\ \downarrow \mathsf{p}\\ \mathbf{Set}\end{array}$ (HJP:80,Pitts:81)

③ $\mathbf{Mod}(A) \rightarrowtail \mathbf{Asm}(A)$ (Hyland:82)

④ $(\mathbf{PartAsm}(A))_{\mathsf{ex/lex}} \simeq (\mathbf{Asm}(A))_{\mathsf{ex/reg}} \simeq \mathbf{Set}[\mathsf{p}] = \mathrm{RT}(A)$ (CFS:87,RR:90)

Typed (Predicative) Realizability

① WCPC-category $\mathbb{C}$

② real. pretripos $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow \mathsf{p}\\ \mathbf{Set}\end{array}$

③ $\mathbf{Mod}(\mathbb{C}) \rightarrowtail \mathbf{Asm}(\mathbb{C})$

④ $(\mathcal{F}(\mathbb{C}))_{\mathsf{ex/lex}} \simeq (\mathbf{Asm}(\mathbb{C}))_{\mathsf{ex/reg}} \simeq \mathbf{Set}[\mathsf{p}]$
(not necessarily a topos)

# Main Results

① $\mathbf{Mod}(\mathbb{C})$ and $\mathbf{Asm}(\mathbb{C})$ model dependent predicate logic (DPL)

Consequence: $\mathbf{Equ}$ models DPL

② $(\mathcal{F}(\mathbb{C})\,)_{\text{ex/lex}} \simeq (\mathbf{Asm}(\mathbb{C})\,)_{\text{ex/reg}} \simeq \mathbf{Set}[\mathrm{p}]$ is a topos iff $\mathbb{C}$ has a universal object

Consequence: $\mathbf{Equ}$ is not a full subcategory of a realizability topos over algebraic lattices

③ Realizability over a $\mathrm{PCA}$ $A$ is the same as realizability over induced WCPC-category $\mathbb{C}(A)$.

# Dependent Predicate Logic (DPL)

DPL = DTT + predicate logic:

- $\Gamma \vdash \sigma : \mathsf{Type}, \quad \Gamma \vdash M : \sigma$
- $\Gamma \vdash \varphi : \mathsf{Prop}, \quad \Gamma \mid \varphi \vdash \psi$
- quotient types $\Gamma \vdash \sigma/\varphi : \mathsf{Type}$
- subset types $\Gamma \vdash \{x : \sigma \mid \varphi\} : \mathsf{Type}$

# $p$-categories

Definitions:

- A $p$**-category** is a category $\mathbb{C}$ endowed with
  - bifunctor $\times\colon \mathbb{C}\times\mathbb{C}\to\mathbb{C}$ (**product**),
  - natural transformation $\Delta\colon (-)\to(-\times-)$ (**diagonal**),
  - two families of natural transformations
    $\{\, p_{-,Y}\colon (-\times Y)\to(-) \mid Y\in\mathbb{C}\,\}$ and
    $\{\, q_{X,-}\colon (X\times -)\to(-) \mid X\in\mathbb{C}\,\}$ (**projections**),

  satisfying some identities.
- Given a map $f\colon X\to Y$ in the $p$-category $\mathbb{C}$, the **domain** $\operatorname{dom} f\colon X\to Y$ of $f$ is the composite map $p_{X,Y}(id_X\times f)\Delta_X\colon X\to X$.
- A map $f\colon X\to Y$ in a $p$-category $\mathbb{C}$ is **total** if $\operatorname{dom} f = id$. Total maps form a subcategory $\mathbb{C}_t$ with finite products.
- A $p$-category $\mathbb{C}$ is **partial cartesian category** if $\mathbb{C}_t$ has a terminal object (hence finite products).

Definition: Let $\mathbb{C}$ be a partial cartesian category. $\mathbb{C}$ is **weakly closed** if, for all $X, Z \in \mathbb{C}$, there exists $[X \rightharpoonup Z]$ and $e\colon [X \rightharpoonup Z] \times X \to Z$ such that, for all $Y$, all $f : Y \times X \to Z$, there exists an $f' : Y \to [X \rightharpoonup Z]$ in $\mathbb{C}_t$ (note that $f'$ is total!), such that:

$$\begin{tikzcd}
{[X \rightharpoonup Z] \times X} \arrow[r, "e"] & Z \\
Y \times X \arrow[u, "f' \times id_X"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Examples: $\mathbf{Ptl}$, $\mathbf{ALat}$ and categories induced by partial combinatory algebras.

# Realizability Pretriposes

Let

- $\mathbb{C}$ be a $\mathrm{WCPC}$-category
- $U : \mathbb{C} \to \mathbf{Ptl}$ be a $p$-functor

Define preordered fibred category $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \big\downarrow \mathsf{p}\\ \mathbf{Set}\end{array}$ with

- fibre over $I$:
  - pairs of the form $(A, \varphi)$, where $A \in \mathbb{C}$, $\varphi\colon I \to P(UA)$ in $\mathbf{Set}$
  - $(A, \varphi) \vdash (B, \psi)$ if and only if

$$\exists g \in \mathbb{C}(A, B).\ \forall i \in I.\ \forall a \in \varphi(i).\ U(g)(a)\downarrow \text{ and } U(g)(a) \in \psi(i).$$

- reindexing by composition.

It is a pretripos (with disjunction if $\mathbb{C}$ has weak coproducts).

# Realizability Pretriposes and Universal Objects

Definition:

A universal object in a category $\mathbb{C}$ is an object $V$ such that all other objects in $C$ are retracts of $V$.

Theorem:

Let $\mathbb{C}$ be a $\mathrm{WCPC}$-category and suppose $U : \mathbb{C} \to \mathbf{Ptl}$ is a faithful $p$-functor. Let $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow \mathrm{p} \\ \mathbf{Set}\end{array}$ be the induced realizability pretripos. Then $\mathbb{C}$ has a universal object if and only if $\mathbf{Set}[\mathrm{p}]$ is a topos.

# Proof Outline, I

We use Pitts's result:

> $\mathbf{Set}[\mathsf{p}]$ is a topos iff axiom (CA) holds.

Axiom (CA):

> For all sets $X$ there is a set $P_X$ and a predicate $In_X \in \mathbf{UFam}(\mathbb{C})_{X \times P_X}$ such that, for any set $I$ and $R \in \mathbf{UFam}(\mathbb{C})_{X \times I}$, the sentence
>
> $$\forall i \colon I.\ \exists s \colon P_X.\ \forall x \colon X.\ \mathrm{In}_X(x, s) \supset\!\subset R(x, i)$$
>
> is satisfied in $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$.

# Proof Outline, II

Suppose $\mathbf{UFam}(\mathbb{C}) \to \mathbf{Set}$ satisfies (CA).

Let

- $X = 1$
- $\Sigma = P_X$
- $V \in \mathbb{C}$ be the underlying object of $\mathrm{In} = \mathrm{In}_1$

Claim: $V$ is universal object in $\mathbb{C}$.

Let $A \in \mathbb{C}$ be arbitrary, let $I = U A$ and let $R(a) = \{a\}$. By (CA) the following sentence is valid in the realizability pretripos:

$$\forall a\colon U A.\ \exists s\colon \Sigma.\ \mathrm{In}(s) \supset\subset R(a).$$

Thus there is a realizer in

$$\bigcap_{a \in U A} \left( \bigcup_{s \in \Sigma} \mathrm{In}(s) \supset\subset R(a) \right) \subseteq U([V \rightharpoonup A] \times [A \rightharpoonup V]).$$

# Related Work

- de Marchi, Robinson, and Rosolini: when is $(\mathcal{F}(\mathbb{C}))_{\text{ex/lex}}$ a topos (for $\mathbb{C}$ not necessarily a WCPC-category) ?
- Longley: PCTS, typed version of PCA
- Lietz and Streicher: PCTS gives rise to a topos iff it has a universal type iff equivalent (in suitable sense) to a PCA.