Developing Theories of Types and Computability via Realizability

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Overview

$$
\begin{tikzcd}
\text{Std. Math. Spaces} \arrow[r, hook] & \text{(Realizability) Models} \arrow[dl] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & \text{Domain Theory} \arrow[l, hook'] \\
\text{Computability} & \text{Denotational Semantics} & \text{Type Theory / Logic}
\end{tikzcd}
$$

Focus of my work Type Theory / Logic, classes of models, foundation for more concrete studies of specific models

Techniques used categorical logic and categorical type theory

Outline

- From a concrete model, combining domain theory and std. math. spaces, to a general notion of realizability
- From general results about general notion of realizability we will obtain some properties of the concrete model
- Suggest a way to incorporate computability into the concrete model
- Leads to relative notion of realizability which accomodates std. math. spaces, computability and domain theory
- Results about relative notion of realizability, in particular, modal logic for computability.

Equilogical Spaces

Category **Equ** with

objects topological T_0-spaces with arbitrary equivalence relations

morphisms equivalence classes of continuous functions preserving the equivalence relations

Motivation:

- contains many categories of domains as used in denotational semantics
- contains many ordinary mathematical spaces, so allows study of computable analysis

$\mathbf{Equ} \simeq \mathbf{PER}(\mathbf{ALat}) \simeq \mathbf{Mod}(\mathbf{ALat})$

PER(ALat):

objects $(A, \equiv_A)$ with A an algebraic lattice and $\equiv_A$ a partial equivalence relations on the underlying set of A

morphisms $(A, \equiv_A) \to (B, \equiv_B)$ are equivalence classes of continous functions $f\colon A \to B$ in **ALat** preserving the PER's

Mod(ALat):

objects (X, A, E_X) with $X \in \mathbf{Set}$, $A \in \mathbf{ALat}$, and $E_X : X \to P\,A$ such that $E_X(x) \cap E_X(x') \neq \emptyset \implies x = x'$.

morphisms $(X, A, E_X) \to (Y, B, E_Y)$ are functions $f\colon X \to Y$ in **Set** such that there exists a $g\colon A \to B$ in **ALat** which realizes f:

$$\forall x \in X.\ \forall a \in E_X(x).\ g(a) \in E_Y(fx)$$

General Notion of Realizability, I

Based on WCPC-category — think of as

- objects are sets of realizers
- morphisms are realizers: a kind of possibly partial functions
- can form the product of sets of realizers
- for any two sets of realizers X and Y there is a set of realizers $[X \rightharpoonup Y]$ which contains the set of realizers from X to Y

General Notion of Realizability, II

Un(i)typed (Impredicative) Realizability

1. PCA A
2. real. tripos $\begin{array}{c}\mathbf{UFam}(A)\\ \downarrow \\ \mathbf{Set}\end{array}$ [HJP:80,Pitts:81]
3. $\mathbf{PER}(A) \simeq \mathbf{Mod}(A) \rightarrowtail \mathbf{Asm}(A)$ [Hyland:82]
4. $(\mathbf{Asm}(A))_{\mathrm{ex/reg}} \simeq \mathrm{RT}(A)$ [CFS:87,RR:90]

Typed (Predicative) Realizability

1. WCPC-category $\mathbb{C}$
2. real. pretripos $\begin{array}{c}\mathbf{UFam}(\mathbb{C})\\ \downarrow \\ \mathbf{Set}\end{array}$
3. $\mathbf{PER}(\mathbb{C}) \simeq \mathbf{Mod}(\mathbb{C}) \rightarrowtail \mathbf{Asm}(\mathbb{C})$
4. $(\mathbf{Asm}(\mathbb{C}))_{\mathrm{ex/reg}}$ (not necessarily a topos)

Main Results

1. $\mathbf{Mod}(\mathbb{C})$ and $\mathbf{Asm}(\mathbb{C})$ model dependent predicate logic (DPL)

   Consequence: **Equ** models DPL

2. $(\mathbf{Asm}(\mathbb{C})\,)_{\mathrm{ex/reg}}$ is a topos iff $\mathbb{C}$ has a universal object

   Consequence: **Equ** is not a full subcategory of a realizability topos over algebraic lattices

Dependent Predicate Logic (DPL)

DPL = DTT + predicate logic:

- $\Gamma \vdash \sigma : \mathsf{Type}, \quad \Gamma \vdash M : \sigma$
- $\Gamma \vdash \varphi : \mathsf{Prop}, \quad \Gamma \mid \varphi \vdash \psi$
- quotient types $\Gamma \vdash \sigma/\varphi : \mathsf{Type}$
- subset types $\Gamma \vdash \{\, x : \sigma \mid \varphi \,\} : \mathsf{Type}$

Relative Realizability and Local Maps of Toposes

Observation:

$$\omega\mathbf{Equ} \simeq \mathbf{PER}(P\omega) \rightarrowtail \mathrm{RT}(P\omega)$$

where $P\omega$ is the graph model of the lambda calc. (a PCA)

Motivation:

- a morphism is computable if it has an *r.e.* realizer — $RE \subseteq P\omega$ is a sub-PCA
- allows to study computable operations on not-necessarily computable data, for example [Bauer], the Riemann integral:

$$I(f,a,b) = \int_a^b f(x)dx :$$
$$: \sharp\big((\mathbb{R} \to \mathbb{R}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\big)$$

Overview and Motivation

- Get subcategory $\mathbf{PER}(P\omega, RE)$. I show that it is a full subcategory of a realizability topos.
- $A_\sharp \subseteq A$
- Examples: Graph Model: $RE \subseteq P\omega$, $\mathrm{TRF} \subseteq \mathbb{N}^{\mathbb{N}}$, term models
- Setup:

$$\begin{tikzcd}
& \mathrm{RT}(A, A_\sharp) \arrow[dl] \arrow[dr] & \\
\mathrm{RT}(A_\sharp) \arrow[ur, shift left=2] \arrow[ur, shift right=2] & & \mathrm{RT}(A)
\end{tikzcd}$$

Outline

- Realizability toposes for computability
- Geometry of realizability toposes for computability
- Axioms for localic local maps
- Modal logic for local maps
- Interpretation in realizability models
- Modal logic example

Definition of $\mathrm{RT}(A, A_\sharp)$

1. Tripos $P\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$
   - non-standard predicates
     $$\varphi, \psi\colon I \to P\,A$$
     and Heyting pre-algebra operations as for tripos over A
   - $\varphi \vdash_I \psi$ iff
     $$\exists a \in A_\sharp.\ \forall i \in I.\ a\colon \varphi(i) \supset \psi(i)$$
2. $\mathrm{RT}(A, A_\sharp) = \mathbf{Set}[P]$
   - Objects: $(X, \approx_X)$, $\approx_X\colon X \times X \to P\,A$, realizers for symmetry and transitivity in $A_\sharp$
   - Morphisms: $f\colon (X, \approx_X) \to (Y, \approx_Y)$ represented by functional relations $F\colon X \times Y \to P\,A$, realizers for functionality in $A_\sharp$
   - Slogan: "continuous objects, computable morphisms"

Geometry of realizability toposes

- Let $Q\colon \mathbf{Set}^{\mathrm{op}} \to \mathbf{Cat}$ be the standard real. tripos on $A_\sharp$
- $\mathbf{Set}$-indexed functors between Q and P:

$$\Delta\colon Q \to P \quad \text{and} \quad \Gamma\colon P \to Q \quad \text{and} \quad \nabla\colon Q \to P$$

- Over 1:
  - $\Delta(\psi \in P\,A_\sharp) = \psi$
  - $\Gamma(\varphi \in P\,A) = A_\sharp \cap \varphi$
  - $\nabla(\psi \in P\,A_\sharp) = \bigcup_{\varphi \in P\,A} \big(\varphi \wedge (A_\sharp \cap \varphi \supset \psi)\big)$
    (where $\wedge$ and $\supset$ are calculated as in $P(1)$)
- Theorem:
  - (Δ, Γ) is a geometric morphism of triposes from P to Q.
  - (Γ, ∇) is a geometric morphism of triposes from Q to P.
  - For all $I \in \mathbf{Set}$, Δ_I and ∇_I are both full and faithful.

At the topos level

$$\begin{tikzcd} \mathrm{RT}(A_\sharp) \arrow[r, bend left, "\Delta"] \arrow[r, bend right, "\nabla"'] & \mathrm{RT}(A, A_\sharp) \arrow[l, "\Gamma"'] \end{tikzcd}, \qquad \Delta \dashv \Gamma \dashv \nabla.$$

- Δ lex.
- Δ and ∇ full and faithful. $\Gamma \circ \nabla \cong 1 \cong \Gamma \circ \Delta$.
- So $(\Delta, \Gamma)\colon \mathrm{RT}(A, A_\sharp) \to \mathrm{RT}(A_\sharp)$ is a connected surjective g.m.
- So $(\Gamma, \nabla)\colon \mathrm{RT}(A_\sharp) \to \mathrm{RT}(A, A_\sharp)$ is an embedding.

$\mathrm{RT}(A_\sharp)$ equivalent to category of sheaves

$$\begin{tikzcd}
& \mathrm{RT}(A, A_\sharp) \arrow[dl, "\Gamma" description] \arrow[dr, "\mathbf{a}" description] & \\
\mathrm{RT}(A_\sharp) \arrow[ur, bend left, "\Delta"] \arrow[ur, bend right, "\nabla"'] \arrow[rr, bend left=10, "\mathbf{a}\,\nabla" description] & & \mathrm{Sh}_j\mathrm{RT}(A, A_\sharp) \arrow[ul, bend right, "i"'] \arrow[ul, bend left, "\Delta\Gamma i"] \arrow[ll, bend left=10, "\Gamma i" description]
\end{tikzcd}$$

$\Delta \dashv \Gamma \dashv \nabla$,

$\Gamma i\,\mathbf{a} \dashv \mathbf{a} \dashv i$,

$\mathbf{a}\,\nabla\Gamma i \cong id$,

$\Gamma i\,\mathbf{a}\,\nabla \cong id$,

Δ f+f and lex,

∇ f+f.

Localic Toposes from Triposes

- Theorem: If $f = (f^*, f_*)\colon P \to Q$ is a geo. mor. of $\mathbb{C}$-triposes, then $\mathbb{C}[P]$ is localic over $\mathbb{C}[Q]$. [Hyland]

  Thus, for all $X \in \mathbb{C}[P]$, there exists $Y \in \mathbb{C}[Q]$ and a diagram

$$\begin{tikzcd} S \arrow[r, rightarrowtail] \arrow[d, twoheadrightarrow] & f^*Y \\ X & \end{tikzcd}$$

  in $\mathbb{C}[P]$ presenting X as a subquotient of f^*Y.

- Thus, $\mathrm{RT}(A, A_\sharp)$ is localic over $\mathrm{RT}(A_\sharp)$ via (Δ, Γ).
- Conclusion: $(\Delta, \Gamma) : \mathrm{RT}(A, A_\sharp) \to \mathrm{RT}(A_\sharp)$ is a **localic local map of toposes**.

Axioms for Localic Local Maps, I

Given topos $\mathcal{E}$ with topology j.

Definitions:

- j is **principal** if, for all $X \in \mathcal{E}$, the closure operation on $\mathrm{Sub}(X)$ has a left adjoint, called **interior**:

$$\mathring{U} \leq V \iff U \leq \overline{V} \qquad \text{in } \mathrm{Sub}(X).$$

- $X \in \mathcal{E}$ is **open** iff $\mathring{X} \cong X$.
- $C \in \mathcal{E}$ is **discrete** if it is **coorthogonal** to all morphisms inverted by $\mathbf{a}$:

$$\begin{tikzcd} & X \arrow[d, "e"] \\ C \arrow[ur, dashed, "f'"] \arrow[r, "f"'] & Y \end{tikzcd} \qquad \mathbf{a}(e) \text{ iso}$$

Axioms for Localic Local Maps, II

Axiom 1. The topology j is principal.

Axiom 2. For all $X \in \mathcal{E}$, there exists a discrete object C and a diagram

$$\begin{tikzcd} S \arrow[r, tail] \arrow[d, two heads] & C \\ X & \end{tikzcd}$$

in $\mathcal{E}$, presenting X as a subquotient of C.

Axiom 3. For all discrete $C \in \mathcal{E}$, if $X \rightarrowtail C$ is open, then X is also discrete.

Axiom 4. For all discrete $C, C' \in \mathcal{E}$, the product $C \times C'$ is again discrete.

Soundness and Completeness

Theorem:

Let $\mathcal{E}$ be a topos with a topology j satisfying the Axioms 1–4 for localic local maps.

The category of discrete objects $\mathrm{D}_j\mathcal{E}$ is coreflective in $\mathcal{E}$, that is, the inclusion $\Delta\colon \mathrm{D}_j\mathcal{E} \to \mathcal{E}$ has a right adjoint. Moreover, $\mathrm{D}_j\mathcal{E}$ is a topos, Δ is left exact, and $(\Delta, \Gamma)\colon \mathcal{E} \to \mathrm{D}_j\mathcal{E}$ is a localic local map.

Theorem:

Every localic local map of toposes satisfies Axioms 1-4 for localic local maps.

A modal logic for local maps, I

Given topos $\mathcal{E}$ with topology j satisfying axioms for localic local maps.

- Goal: Logic to reason about the two toposes $\mathcal{E}$ and $\mathrm{D}_j\mathcal{E}$.
- Problem: interior does not commute with pullback.
- A solution: consider

$$\begin{tikzcd} \mathrm{Pred} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sub}(\mathcal{E}) \arrow[d] \\ \mathrm{D}_j\mathcal{E} \arrow[r, "\Delta"'] & \mathcal{E}. \end{tikzcd}$$

  - types and terms: discrete objects and morphisms between such
  - predicates on type X are all predicates in $\mathcal{E}$ on ΔX

A modal logic for local maps, II

$$\mathrm{Sub}_{\mathcal{E}}(\Delta X) \cong \mathcal{E}(\Delta X, \Omega_{\mathcal{E}}) \cong \mathrm{D}_j\mathcal{E}(X, \Gamma\Omega_{\mathcal{E}})$$

Hence:

- the resulting logic is tripos logic on the internal locale $\Gamma\Omega_{\mathcal{E}}$ in $\mathrm{D}_j\mathcal{E}$.
- the tripos is $H = \mathrm{D}_j\mathcal{E}(-, \Gamma\Omega_{\mathcal{E}})$ and $\mathcal{E} = \mathrm{D}_j\mathcal{E}[H]$.

Interior commutes with pullback in $\begin{array}{c}\mathrm{Pred}\\ \downarrow\\ \mathrm{D}_j\mathcal{E}\end{array}$, so gives a logical operation on predicates; we denote it by $\sharp$.

The topology j in $\mathcal{E}$ induces a closure operator, $\flat$, on predicates in this internal logic.

A modal logic for local maps, III

$$\frac{}{\Gamma \mid \sharp\varphi \vdash \varphi} \qquad \frac{}{\Gamma \mid \sharp\varphi \vdash \sharp\sharp\varphi}$$

$$\frac{}{\Gamma \mid \emptyset \vdash \sharp(T)} \qquad \frac{}{\Gamma \mid \sharp\varphi \wedge \sharp\psi \vdash \sharp(\varphi \wedge \psi)}$$

$$\frac{\Gamma \mid \sharp\varphi \vdash \psi}{\Gamma \mid \varphi \vdash \flat\psi}$$

From this one proves that $\sharp$ has the formal properties of box in S4. We therefore refer to this logic as a **modal logic for computability**. Further:

$$\frac{}{\Gamma \mid \sharp\exists x{:}X.\ \varphi \dashv\vdash \exists x{:}X.\ \sharp\varphi}$$

Internal Locale Viewpoint

Internally in $\mathrm{D}_j\mathcal{E}$:

$$\begin{tikzcd}
\Gamma\Omega_{\mathcal{E}} \arrow[d, "\Gamma" description] \\
\Omega_{\mathrm{D}_j\mathcal{E}} \arrow[u, bend left=60, "\Delta"] \arrow[u, bend right=60, "\nabla"']
\end{tikzcd}
\qquad \Delta \dashv \Gamma \dashv \nabla \qquad \Gamma\Delta = 1 = \Gamma\nabla$$

- $\sharp = \Delta\Gamma$ — has S4 properties automatically
- the further adjoint means that $\sharp$ commutes with existentials

Interpretation in $\mathrm{RT}(A_\sharp)$ and $\mathrm{RT}(A, A_\sharp)$

- types and terms: objects and morphisms of $\mathrm{RT}(A_\sharp)$
- predicates φ on type $(X, \approx)$ are $\varphi \colon X \to P\,A$, strict and extensional via realizers from $A_\sharp$
- $\sharp\varphi(x) = \varphi(x) \cap A_\sharp$
- $\flat\varphi(x) = \bigcup_{\psi \in P\,A}(\psi \wedge (\psi \cap A_\sharp \supset \varphi(x) \cap A_\sharp))$

Modal Logic Example

Church's Thesis:

$$\forall n.\ \exists m.\ \varphi(n,m)) \supset \exists k.\ \forall n.\ \exists m.\ \big(T(k,n,m) \wedge \varphi(n,Um)\big).$$

Valid in RT(RE), not valid in RT($P\omega$), not valid in RT($P\omega, RE$)

Modal Church's Thesis:

$$\sharp(\forall n.\ \exists m.\ |\varphi|(n,m)) \supset \exists k.\ \forall n.\ \exists m.\ \big(T(k,n,m) \wedge |\varphi|(n,Um)\big).$$

valid in modal logic for local map RT($P\omega, RE$) $\to$ RT(RE).

Conclusion

- General Notion of Realizability
- Relative Notion of Realizability, $\mathrm{RT}(A, A_\sharp)$
- Elementary Axioms for Local Maps of Toposes
- Modal Logic for Local Maps of Toposes

Future Work

- Morphisms between WCPC-categories
- Completeness of modal logic for local maps
- Computability and mathematical spaces in concrete models
- Domain theory, recursive domain equations
- Modified relative realizability