The Semantics of Dependent Type Theory

Frederik Lerbjerg Aagaard, 201608140
Aarhus University

24th June 2020

Contents

Chapter 1

Introduction

Type theory is a general term for a wide variety of formal systems. It is closely related to type systems in programming languages but it also has deep ties with category theory, and it can even serve as a foundation of mathematics. The fundamental building blocks of type theory are types and terms, with each term inhabiting exactly one type. In dependent type theory one allows dependent types, which are types that depend on the terms of another type. Categories with families are category theoretical models of dependent type theories, which can be augmented with dependent right adjoints to model modal type theory (type theory with modal operators). A particular such model is the topos of trees, in which it is possible to solve certain equations of types, which allows the recursive definition of types.

In Chapter 2 we make a presentation of a dependent type theory mostly based on [6] including type formers for function, product, Π-, and Σ-types, universes à la Tarski, and modality (the last based on [4] and [3]).

In Chapter 3 we present categories with families as a model of dependent type theory[5] and give two general constructions of them in the form of Giraud CwFs[3] and presheaf CwFs[6]. We also show how to incorporate modality into CwFs making them categories with dependent right adjoints[3].

In Chapter 4 we consider the categories of presheafs on ω, called the topos of trees, and the later modality, both from [2], which makes it a CwDRA. We define contractive morphisms in it, see how they give rise to fixpoints[2], and finish by giving an example of how to solve guarded recursive domain equations.

This project was originally meant to be just 10-20 pages with models of dependent type theory, dependent right adjoints, and the later modality each being supposed to have only a "short explanation" according to the project description. However, as this project is intended to be used by future students, and I do not believe a short explanation would likely leave a future student better off than I have been reading the existing litterature, I have throughout given considerably more than a short explanation, usually including all non-trivial calculations. My choice of this approach is heavily influenced by my own preference in learning mathematics, as I personally feel I understand mathematics much better once I have seen the details, either through reading them or figuring them out on my own. Naturally, these details are accompanied by explanations of the useful intuitions I have acquired through my own learning process. Due to the longer than planned nature of this project, several things that I feel should have been included have not been for reasons of time. Of particular note is the explicit construction of type formers in presheaf CwFs, an exploration of when endofunctors on presheaf CwFs extend to weak CwF morphisms, and locally contractive functors and their fixpoints, which allows a general method of solving equations like that in Section 4.4. I plan on adding these at a later date for the further benefit of future students.

Chapter 2

Type Theory

Type theory is at its core a foundation for mathematics, just like set theory, and just like set theory the term covers many different versions. At the heart of any such foundation lies judgements and rules. In set theory, we rarely consciously talk about judgements and rules, but we actually do use them all the time. Set theory has one judgement, namely "P `has a proof`", where P is a proposition[1], and the rules are the rules of logic. Here are for example the rules governing $\wedge$:

$$\frac{P \texttt{ has a proof} \qquad Q \texttt{ has a proof}}{P \wedge Q \texttt{ has a proof}}$$

$$\frac{P \wedge Q \texttt{ has a proof}}{P \texttt{ has a proof}} \qquad \frac{P \wedge Q \texttt{ has a proof}}{Q \texttt{ has a proof}}$$

These can be read as a form of implication with the judgement(s) above the line "implying" the judgements below it. It is however not really implication as judgements do not have a truth value; they are ether valid or they are not judgements at all.

In set theory we have sets (or classes), and the propositions come from first order logic, whilst in type theory we have contexts, types, and terms, but we have no propositions. Instead of propositions, we can in type theory interpret types as propositions, by saying that a type is true if a term has it as its type.

The judgements of type theory can very quite a bit depending on the details, but we will adopt the judgements of [6]:

$\vdash \Gamma$ `ctxt`	Γ is a context
$\Gamma \vdash A$ `type`	A is a type in context Γ
$\Gamma \vdash a : A$	a is a term of type A in context Γ
$\vdash \Gamma \equiv \Delta$ `ctxt`	Γ and Δ are judgementally equal contexts
$\Gamma \vdash A \equiv B$ `type`	A and B are judgementally equal types in context Γ
$\Gamma \vdash a \equiv b : A$	a and b are judgementally equal terms of type A in context Γ

Note that some of these judgements involve some presuppositions, e.g. $\Gamma \vdash A$ `type` presupposes that $\vdash \Gamma$ `ctxt`, and $\Gamma \vdash a \equiv b : A$ presupposes that $\vdash \Gamma$ `ctxt`, $\Gamma \vdash A$ `type`, $\Gamma \vdash a : A$, and $\Gamma \vdash b : A$. The others are similar.

We will first consider the rules of formation of contexts. The first rule simply states, there exists a context:

$$\frac{}{\vdash \diamond \texttt{ ctxt}}$$

[1]Here proposition means statement rather than proven statement.

This reads: with no assumptions, $\diamond$ is a context. The next two rules deal with extension of contexts and variables:

$$\frac{\Gamma \vdash A \;\; \texttt{type}}{\vdash \Gamma, x : A \;\; \texttt{ctxt}} \qquad \frac{\vdash \Gamma, x : A, \Delta \;\; \texttt{ctxt}}{\Gamma, x : A, \Delta \vdash x : A}$$

The x above is called a variable. When working with variables, we need to be careful with naming, as we should not have two variables with the same name. The rule on the left therefore assumes that no other variable of Γ is called x, i.e. that x is a fresh variable. If we combine these two rules, we get the following deduction

$$\cfrac{\cfrac{\Gamma \vdash A \;\; \texttt{type}}{\vdash \Gamma, x : A \;\; \texttt{ctxt}}}{\Gamma, x : A \vdash x : A}$$

This highlights the importance of the context we are working in, as this tells us that given any context Γ and any type A in this context, we can find a term x of type A in the context $\Gamma, x : A$, but crucially not necessarily in the context Γ. Thus one can understand $\Gamma, x : A$ artificially adding a term of type A even if no such term actually exists in the context Γ.

We also have rules stating that if we replace on part of a judgement with an equal object, nothing really changes:

$$\frac{\vdash \Gamma \equiv \Delta \;\; \texttt{ctxt} \qquad \Gamma \vdash A \;\; \texttt{type}}{\Delta \vdash A \;\; \texttt{type}} \qquad \frac{\vdash \Gamma \equiv \Delta \;\; \texttt{ctxt} \qquad \Gamma \vdash A \equiv B \;\; \texttt{type} \qquad \Gamma \vdash a : A}{\Delta \vdash a : B}$$

$$\frac{\vdash \Gamma \equiv \Delta \;\; \texttt{ctxt} \qquad \Gamma \vdash A \equiv B \;\; \texttt{type}}{\Delta \vdash A \equiv B \;\; \texttt{type}} \qquad \frac{\vdash \Gamma \equiv \Delta \;\; \texttt{ctxt} \qquad \Gamma \vdash A \equiv B \;\; \texttt{type} \qquad \Gamma \vdash a \equiv b : A}{\Delta \vdash a \equiv b : B}$$

$$\frac{\vdash \Gamma \equiv \Delta \;\; \texttt{ctxt} \qquad \Gamma \vdash A \equiv B \;\; \texttt{type}}{\vdash \Gamma, x : A \equiv \Delta, y : B \;\; \texttt{ctxt}}$$

There are also rules stating that each of the equalities is reflexive, symmetric, and transitive, but we leave this to the reader. Next we add the rule of weakening, which states that context comprehension does not remove any types, terms, or equalities:

$$\frac{\Gamma, \Delta \vdash \mathcal{J} \qquad \Gamma \vdash A \;\; \texttt{type}}{\Gamma, x : A, \Delta \vdash \mathcal{J}}$$

Here $\mathcal{J}$ is either B $\texttt{type}$, $B \equiv C$ $\texttt{type}$, $b : B$, or $b \equiv c : B$. This is a so-called admissible rule, which means that without assuming the rule, if the premisses are deducible, then so is the conclusion. In that sense, it is not strictly necessary to add as a rule, but on the other hand it saves quite a bit of work and nothing is really lost, which is why we happily add it.

The last thing to tackle is substitution. Whilst we have not seen how to do so yet, terms and types can be large expressions which include variables, and substitution is replacing that variable with a term of the appropriate type. We actually do this all the time in set theory: Consider the expression $x + 1$, where $x \in \mathbb{Z}$. If we set $x = 1$, then $x + 1 = 1 + 1$, which is exactly the same as just replacing all occurrences of x by 1. We denote the substitution of the variable $x : A$ by the term $a : A$ by $[a/x]$, e.g. $(x + 1)[1/x] = 1 + 1$. This is encompassed in the substitution rule

$$\frac{\Gamma, x : A, \Delta \vdash \mathcal{J} \qquad \Gamma \vdash a : A}{\Gamma, \Delta[a/x] \vdash \mathcal{J}[a/x]}$$

Here $\mathcal{J}$ once again ranges over B $\texttt{type}$, $B \equiv C$ $\texttt{type}$, $b : B$, or $b \equiv c : B$. With all the fundamental rules covered, we will now move on the type formers.

2.1 Function Types

A type former is a way to use terms and types to construct new types with certain properties. In order to define a type former one (generally) needs to define 4-5 things:

- Formation rules, stating how to form the type.
- Introduction rules, stating how to make terms of the type.
- Elimination rules, stating how to use terms of the type.
- Computation rules, stating how the elimination rules act introduced terms.
- Uniqueness rules (optional), stating that every term is given by a particular form (usually the introduction rules applied to eliminated terms).

We will describe a number of types, starting with function types. Given two types A and B, we wish to define a type $A \to B$, whose terms behave like functions from A to B. We have here already described the formation rule

$$\frac{\Gamma \vdash A, B \ \texttt{type}}{\Gamma \vdash A \to B \ \texttt{type}}$$

where $\Gamma \vdash A, B \ \texttt{type}$ is shorthand for $\Gamma \vdash A \ \texttt{type}$ and $\Gamma \vdash B \ \texttt{type}$. For the introduction rule, we will need some way of making functions. In set theory, we would write something along the lines of

$$\begin{aligned} f \colon A &\to B \\ x &\mapsto b, \end{aligned}$$

where b is some expression possibly depending on x taking values in B. An alternative notation for this originating in computer science is $\lambda x.b$, which means the function taking x to b, and this will be our chosen notation in the introduction rule:

$$\frac{\Gamma \vdash A, B \ \texttt{type} \qquad \Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda(x : A).b : A \to B}$$

For the elimination rule, we will need some way of turning a function into not a function, which is exactly what function application does:

$$\frac{\Gamma \vdash A, B \ \texttt{type} \qquad \Gamma \vdash f : A \to B \qquad \Gamma \vdash a : A}{\Gamma \vdash f(a) : B}$$

For the computation rule, we need to describe how to apply the elimination rule to an introduced term. We described earlier that $\lambda x.b$ takes x to b, and thus have our description of how λ expressions act under function application:

$$\frac{\Gamma \vdash A, B \ \texttt{type} \qquad \Gamma, x : A \vdash b : B \qquad \Gamma \vdash a : A}{\Gamma \vdash (\lambda(x : A).b)(a) \equiv b[a/x] : B}$$

Finally for uniqueness, the equivalent statement for set theory is that $f = g$ if and only if $f(x) = g(x)$ for all x. Traditionally, type theory uses a slightly different but equivalent formulation, namely that $f = \lambda x.f(x)$[2]:

$$\frac{\Gamma \vdash A, B \ \texttt{type} \qquad \Gamma \vdash f : A \to B}{\Gamma \vdash f \equiv \lambda(x : A).f(x) : A \to B}$$

[2]Why are these statements equivalent?

2.2 Product Types

The product type is the type theoretic version of $A \times B$, and it is thus clear that we have the formation rule

$$\frac{\Gamma \vdash A, B \ \texttt{type}}{\Gamma \vdash A \times B \ \texttt{type}}$$

In set theory an element of $A \times B$ is a pair (a, b), where $a \in A$ and $b \in B$, and this will be our introduction rule

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash (a, b) : A \times B}$$

In set theory, we can take either the first or the second coordinate of $A \times B$ to get elements of A respectively B, and this will be our elimination rules

$$\frac{\Gamma \vdash A, B \ \texttt{type} \qquad \Gamma \vdash t : A \times B}{\Gamma \vdash p_1(t) : A} \qquad \frac{\Gamma \vdash A, B \ \texttt{type} \qquad \Gamma \vdash t : A \times B}{\Gamma \vdash p_2(t) : B}$$

Note that p_1 and p_2 are thus far purely symbolic rather than representing functions, but one can of course define functions $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$ that do exactly as above.

For the computation rule, we need to see how elimination interacts with introduction, i.e. what the first respectively second coordinate of (a, b) is, which should be obvious

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash p_1(a, b) \equiv a : A} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B}{\Gamma \vdash p_2(a, b) \equiv b : B}$$

The next step is to add a uniqueness rule, but it turns out that this is unnecessary, as one can actually derive the rule. We will not go into this here, but it can be found in [8, Section 1.5].

2.3 Universes

We will now take a brief break from the type formers in order to consider universes. One limitation that type theory has is that terms and types are fundamentally different things, and they do not mix. In set theory it is unproblematic to define a function from a set A, which can take values anywhere, but in type theory we have defined thus far there is no way to define a function from a type whose values are types. This is the problem that universes solve. There are several approaches to universes; we will here present universes à la Tarski (of Banach-Tarski paradox fame).

No matter the version of the universes, the general idea is to have a "type of types". In universes à la Tarski this is accomplished by having a type, whose terms in some way corresponds to types. The setup with regards to rules is quite similar to earlier, but we only have two rules, formation and elimination. These are

$$\frac{\vdash \Gamma \ \texttt{ctxt}}{\Gamma \vdash U \ \texttt{type}} \qquad \frac{\Gamma \vdash A : U}{\Gamma \vdash \mathrm{El}(A) \ \texttt{type}}$$

These state that we have a universe in any context, and that we can convert a term of a universe into a type, which is called the elements of the term. With only these rules, universes are somewhat lacklustre, as they lack any of the structure that we have previously given types with type formers. We therefore add additional rules to ensure that universes are stable under type formers and El respects them. There are several ways of achieving this, but we choose the following for its simplicity:

$$\frac{\Gamma \vdash A, B : U}{\Gamma \vdash A \to B : U} \qquad \frac{\Gamma \vdash A, B : U}{\Gamma \vdash \mathrm{El}(A \to B) \equiv \mathrm{El}(A) \to \mathrm{El}(B) \ \ \texttt{type}}$$

$$\frac{\Gamma \vdash A, B : U}{\Gamma \vdash A \times B : U} \qquad \frac{\Gamma \vdash A, B : U}{\Gamma \vdash \mathrm{El}(A \times B) \equiv \mathrm{El}(A) \times \mathrm{El}(B) \ \ \texttt{type}}$$

Because of this need to ensure that universes respect type formers, we will when introducing new type formers need to also add rules like the above.

2.4 Dependent Functions

Dependent functions are a generalization of functions, which is quite common in set theory though we only rarely think of it as a function. A normal function from A to B takes an element of A to an element of B, but imagine now instead that B is a function from A that takes as values any sets; then we can imagine a function from A to B, such that each $a \in A$ is mapped to an element of $B(a)$. This is a dependent function.

One place where we actually meet functions like this in set theory, though we don't call it a dependent function, is the axiom of choice, which states that given any set of non-empty sets, call it A, there exists a function from A, which takes each set to an element of itself; in other words there exists a dependent function from A to Id_A.

A more general case of dependent functions is indexed Cartesian products, i.e. things like

$$\prod_{a \in A} B(a).$$

We rarely think of this set as consisting of functions, but consider what information an element of it contains: for each $a \in A$, it knows of an element of $B(a)$, which is exactly what a dependent function is. We will actually be using the above notation for the set (and later, type) of dependent functions.

Let us consider for a moment the case, where B is constantly equal to some set, which we will call B'. In this case a dependent function from A to B is the same thing as a function from A to B'. Using the above notation, the set of all dependent functions from A to B is

$$\prod_{a \in A} B(a) = \prod_{a \in A} B',$$

and the set of all functions from A to B' is usually denoted, B'^A, and we thus see that the two notations fit perfectly together as $\prod_{a \in A} B'$ should be the same as B'^A.

Now on to the type theory. We will also here adopt the notation from above[3]. The reader should contrast and compare the rules to the rules for function types. The formation rule is

$$\frac{\Gamma \vdash A \ \ \texttt{type} \qquad \Gamma, x : A \vdash B \ \ \texttt{type}}{\Gamma \vdash \Pi_{(x:A)} B \ \ \texttt{type}}$$

Remember that $\Gamma, x : A \vdash B$ `type` means that B may depend on the variable x, and we thereby have the type theoretic version of a function from A that takes sets as values. At the end of this section, we will see how to use universe to make Π-types with terms of function types. The rest of the rules are

$$\frac{\Gamma \vdash A \ \ \texttt{type} \qquad \Gamma, x : A \vdash b : B}{\Gamma \vdash \lambda(x : A).b : \Pi_{(x:A)} B} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma, x : A \vdash B \qquad \Gamma \vdash f : \Pi_{(x:A)} B}{\Gamma \vdash f(a) : B[a/x]}$$

[3]Because of this notation, dependent function types are also called Π-types

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A \vdash b : B}{\Gamma \vdash (\lambda(x:A).b)(a) \equiv b[a/x] : B[a/x]} \qquad \frac{\Gamma \vdash A \ \texttt{type} \qquad \Gamma, x : A \vdash B \qquad \Gamma \vdash f : \Pi_{(x:A)} B}{\Gamma \vdash f \equiv \lambda(x:A).f(x) : \Pi_{(x:A)} B}$$

The rules for universes are

$$\frac{\Gamma \vdash A : U \qquad \Gamma, x : \mathrm{El}(A) \vdash B : U}{\Gamma \vdash \Pi_{(x:A)} B : U} \qquad \frac{\Gamma \vdash A : U \qquad \Gamma, x : \mathrm{El}(A) \vdash B : U}{\Gamma \vdash \mathrm{El}(\Pi_{(x:A)} B) \equiv \Pi_{(x:\mathrm{El}(A))} \mathrm{El}(B) \ \texttt{type}}$$

In the above presentation of Π-types, B is not really a function as was the case in set theory but rather a type, which depends on a variable of type A. Using universes, we can however remedy this, as a function $A \to U$ sends terms of A to terms of U, which through El can be considered types. Intuitively, we should this have the following rule:

$$\frac{\Gamma \vdash A \ \texttt{type} \qquad \Gamma \vdash B : A \to U}{\Gamma \vdash \Pi_{(x:A)} \mathrm{El}(B(x)) \ \texttt{type}}$$

The reason we do not give this as the definition is that we have no guarantee that the universe contains all types, and therefore not all Π-types are given by the above rule. This rule can actually be derived quite easily. We have not really talked much about derivation yet, as it is not of much relevancy to our goals, but briefly, a derivation is a successive application of rules to some premisses until a conclusion is reached. The derivation of the above rule goes as follows:

$$\dfrac{\Gamma \vdash A \ \texttt{type} \qquad \dfrac{\dfrac{\dfrac{\Gamma \vdash A \ \texttt{type} \qquad \dfrac{\vdash \Gamma \ \texttt{ctxt}}{\Gamma \vdash U \ \texttt{type}}}{\Gamma, x : A \vdash A, U \ \texttt{type}} \qquad \dfrac{\Gamma \vdash B : A \to U}{\Gamma, x : A \vdash B : A \to U} \qquad \dfrac{\dfrac{\Gamma \vdash A \ \texttt{type}}{\vdash \Gamma, x : A \ \texttt{ctxt}}}{\Gamma, x : A \vdash x : A}}{\Gamma, x : A \vdash B(x) : U}}{\Gamma, x : A \vdash \mathrm{El}(B(x)) \ \texttt{type}}}{\Gamma \vdash \Pi_{(x:A)} \mathrm{El}(B(x)) \ \texttt{type}}$$

One can get similar rules corresponding to introduction, elimination, computation, and uniqueness:

$$\frac{\Gamma \vdash A \ \texttt{type} \qquad \Gamma \vdash B : A \to U \qquad \Gamma, x : A \vdash b : \mathrm{El}(B(x))}{\Gamma \vdash \lambda(x:A).b : \Pi_{(x:A)} \mathrm{El}(B(x))}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash B : A \to U \qquad \Gamma \vdash f : \Pi_{(x:A)} \mathrm{El}(B(x))}{\Gamma \vdash f(a) : B(a)}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash B : A \to U \qquad \Gamma, x : A \vdash b : \mathrm{El}(B(x))}{\Gamma \vdash (\lambda(x:A).b)(a) \equiv b[a/x] : B(a)}$$

$$\frac{\Gamma \vdash A \ \texttt{type} \qquad \Gamma \vdash B : A \to U \qquad \Gamma \vdash f : \Pi_{(x:A)} \mathrm{El}(B(x))}{\Gamma \vdash f \equiv \lambda(x:A).f(x) : \Pi_{(x:A)} \mathrm{El}(B(x))}$$

2.5 Dependent Pairs

In set theory, the graph of a function $f\colon A \to B$ is a subset of $A \times B$, but what is the graph of a dependent function $f \in \prod_{a\in A} B(a)$ a subset of? Each element of it should be a pair (a, b), where $b \in B(a)$, and we thus have the idea a dependent pair, which gives rise to the definition

$$\sum_{a\in A} B(a) = \bigcup_{a\in A} (\{a\} \times B(a)).$$

Much like Π-types were a generalization of function types, dependent pair types[4] are a generalization of product types. We will here simply state the rules, but we invite the reader to compare them to the rules of product types.

$$\frac{\Gamma \vdash A \ \texttt{type} \qquad \Gamma, x : A \vdash B \ \texttt{type}}{\Gamma \vdash \Sigma_{(x:A)} B \ \texttt{type}} \qquad \frac{\Gamma \vdash a : A \qquad \Gamma, x : A \vdash B \ \texttt{type} \qquad \Gamma \vdash b : B[a/x]}{\Gamma \vdash (a, b) : \Sigma_{(x:A)} B}$$

$$\frac{\Gamma \vdash A \ \texttt{type} \qquad \Gamma, x : A \vdash B \ \texttt{type} \qquad \Gamma \vdash t : \Sigma_{(x:A)} B}{\Gamma \vdash p_1(t) : A}$$

$$\frac{\Gamma \vdash A \ \texttt{type} \qquad \Gamma, x : A \vdash B \ \texttt{type} \qquad \Gamma \vdash t : \Sigma_{(x:A)} B}{\Gamma \vdash p_2(t) : B[a/x]}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A \vdash B \ \texttt{type} \qquad \Gamma \vdash b : B[a/x]}{\Gamma \vdash p_1(a, b) \equiv a : A}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma, x : A \vdash B \ \texttt{type} \qquad \Gamma \vdash b : B[a/x]}{\Gamma \vdash p_2(a, b) \equiv b : B[a/x]}$$

$$\frac{\Gamma \vdash A : U \qquad \Gamma, x : \mathrm{El}(A) \vdash B : U}{\Gamma \vdash \Sigma_{(x:A)} B : U} \qquad \frac{\Gamma \vdash A : U \qquad \Gamma, x : \mathrm{El}(A) \vdash B : U}{\Gamma \vdash \mathrm{El}(\Sigma_{(x:A)} B) \equiv \Sigma_{(x:\mathrm{El}(A)} \mathrm{El}(B) \ \texttt{type}}$$

2.6 Modality

The concept of modality stems from linguistics, where it allows the expression of intensions as well as belief of propositions. In English this is done via the modal verbs, e.g. "I *will* finish this tonight" and "It *might* rain tomorrow". From this linguistic notation we get the related notion in logic of modal operators, which are operators on propositions, modelling things like necessity and possibility. These modal operators are usually written $\Box$ and $\diamond$ respectively.

In 1994 Borghuis presented in his ph.d. thesis[4] various type theories modelling modal logic by using a propositions-as-types interpretation, i.e. having types correspond to propositions. Because of this correspondence, we must have some operator on types $\Box$, though its rules will be established later. As part of the proof system of the logic used is the concept of a strict subordinate proof. Usually, one is allowed in a subordinate proof to use any previously established propositions, but in a strict subordinate proof one is only allowed to use a proposition P if $\Box P$ has previously been established, and conversely if P is proven in a strict subordinate proof $\Box P$ may be used in the main proof. In the type theory this is handled by letting the context contain the previously established information, and then applying an operator to the context, which locks away any information not of the form $\Box P$. Using the notation of [3], where 🔒[5] is the operator on contexts, we thus have the rules

[4] Also called Σ-types due to the notation

[5] This symbol is written with the `\faLock` command in the `fontawesome` package.

$$\frac{\vdash \Gamma \ \ \texttt{ctxt}}{\vdash \Gamma, \text{🔒} \ \ \texttt{ctxt}} \qquad \frac{\Gamma, \text{🔒} \vdash A \ \ \texttt{type}}{\Gamma \vdash \Box A \ \ \texttt{type}}$$

We mention also the admissible rule[6]

$$\frac{\Gamma \vdash \Box A \ \ \texttt{type}}{\Gamma, \text{🔒} \vdash A \ \ \texttt{type}}$$

and thus if we view these as the rules of a context former, we have respectively formation, elimination, and introduction, though we can also view the second rule as the formation tuel for $\Box$. In order to deal with terms, we introduce `shut` and `open` with the following introduction and elimination rules for $\Box$

$$\frac{\Gamma, \text{🔒} \vdash a : A}{\Gamma \vdash \texttt{shut}\, a : \Box A} \qquad \frac{\Gamma \vdash a : \Box A \qquad \vdash \Gamma, \text{🔒}, \Delta \ \ \texttt{ctxt}}{\Gamma, \text{🔒}, \Delta \vdash \texttt{open}\, a : A}$$

where we in the second rule suppose that 🔒 does not appear in Δ. Finally, we also have computation and uniqueness rules

$$\frac{\Gamma \vdash \texttt{open}\, \texttt{shut}\, a : A}{\Gamma \vdash \texttt{open}\, \texttt{shut}\, a \equiv a : A} \qquad \frac{\Gamma \vdash a : \Box A}{\Gamma \vdash a \equiv \texttt{shut}\, \texttt{open}\, a : \Box A}$$

Note that was this set theory, we would say that the last four rules state that `shut` and `open` are mutually inverse bijections between the terms of A and the terms of $\Box A$.

[6] Recall that a rule is admissible if, whenever the premisses are deducible, so is the conclusion

Chapter 3

Categories with Families

3.1 The Category of Elements

Before we can introduce categories with families, we must acquaint ourselves with the category of elements of a presheaf.

Definition 3.1. Let $\mathbf{C}$ be a locally small category, and let $P\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf on $\mathbf{C}$. We then define the category of elements of P, denoted $\int_{\mathbf{C}}(P)$, as follows:

- The objects are tuples (X, a), where $X \in \mathbf{C}_0$ and $a \in P(X)$, i.e.

$$\int_{\mathbf{C}}(P)_0 = \sum_{X \in \mathbf{C}_0} P(X).$$

- The morphisms from (X, a) to (Y, b) are pairs (f, b), where $f\colon X \to Y$ is a morphism with $P(f)(b) = a$, i.e.

$$\mathrm{Hom}_{\int_{\mathbf{C}}(P)}((X,a),(Y,b)) = \{(f,b) : f \in \mathrm{Hom}_{\mathbf{C}}(X,Y), P(f)(b) = a\}.$$

- Composition of $(f, b)\colon (X, a) \to (Y, b)$ and $(g, c)\colon (Y, b) \to (Z, c)$ is given by

$$(g, c) \circ (f, b) = (g \circ f, c).$$

- The identity morphism on (X, a) is (Id_X, a).

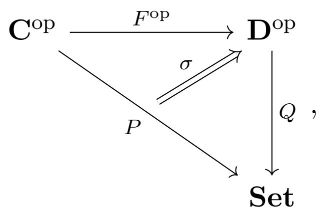
We here use the nomenclature of [7] taking the category of elements of a contravariant functor, whereas others take the category of elements of covariant functors in a completely parallel definition.

We can also use functors on the base categories along with natural transformation to induce functors between the categories of elements:

Definition 3.2. Let $\mathbf{C}$ and $\mathbf{D}$ be locally small categories, let $F\colon \mathbf{C} \to \mathbf{D}$, $P\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, and $Q\colon \mathbf{D}^{\mathrm{op}} \to \mathbf{Set}$ be functors, and let $\sigma\colon P \to Q \circ F^{\mathrm{op}}$ be a natural transformation. We then define the functor

$$\begin{aligned}\int_F(\sigma)\colon \int_{\mathbf{C}}(P) &\to \int_{\mathbf{D}}(Q)\\ (X,a) &\mapsto (F(X), \sigma_X(a))\\ (f,b) &\mapsto (F(f), \sigma_{\mathrm{Cod}_{\mathbf{C}}(f)}(b)).\end{aligned}$$

In the above definition, the starting situation is the diagram



which after application of $\int$ becomes a functor from $\int_{\mathbf{C}}(P)$ to $\int_{\mathbf{D}}(Q)$.

Consider the following category:

- The objects are tuples $(\mathbf{C}, P)$, where $\mathbf{C}$ is a small category, and P is a presheaf on $\mathbf{C}$.
- A morphism from $(\mathbf{C}, P)$ to $(\mathbf{D}, Q)$ is a pair (F, σ), where $F\colon \mathbf{C} \to \mathbf{D}$, and $\sigma\colon P \to Q \circ F^{\mathrm{op}}$.
- Composition of $(F,\sigma)\colon (\mathbf{C}, P) \to (\mathbf{D}, Q)$ and $(G,\tau)\colon (\mathbf{D}, Q) \to (\mathbf{E}, R)$ is defined as

  $$(G,\tau) \circ (F,\sigma) = (G \circ F, (\tau * \mathrm{Id}_F) \circ \sigma),$$

  where $*$ denotes horizontal composition of natural transformations.
- The identity morphism of $(\mathbf{C}, P)$ is $(\mathrm{Id}_{\mathbf{C}}, \mathrm{Id}_P)$.

A simple calculation then shows that $\int_{\_}$ defines a functor from this category to **Cat**.

3.2 Categories with Families

A category with families is a categorical model of type theory. We define it as follows:

Definition 3.3. A category with families is a tuple $(\mathbf{C}, 1, \mathrm{Ty}, \mathrm{Tm}, - \cdot -, \Phi_{-,-})$[1], where $\mathbf{C}$ is a locally small category, $1 \in \mathbf{C}_0$ is a terminal object, $\mathrm{Ty}\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, $\mathrm{Tm}\colon \int_{\mathbf{C}}(\mathrm{Ty})^{\mathrm{op}} \to \mathbf{Set}$, $- \cdot -\colon \int_{\mathbf{C}}(\mathrm{Ty})_0 \to \mathbf{C}_0$, and for all $(\Gamma, A) \in \int_{\mathbf{C}}(\mathrm{Ty})_0$ $\Phi_{\Gamma,A}$ is a natural isomorphism from $\mathrm{Hom}_{\mathbf{C}}(-, \Gamma \cdot A)$ to

$$\begin{aligned}S_{\Gamma,A}\colon \mathbf{C}^{\mathrm{op}} &\to \mathbf{Set}\\ \Delta &\mapsto \sum_{\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta,\Gamma)} \mathrm{Tm}(\Delta, \mathrm{Ty}(\gamma)(A))\\ \delta &\mapsto ((\gamma, a) \mapsto (\gamma \circ \delta, \mathrm{Tm}(\delta, \mathrm{Ty}(\gamma)(A))(a)).\end{aligned}$$

The name "category with families" may seem a bit strange; where are the families? The name comes from the original definition, which is equivalent to the above, which uses so called families of sets. The original definition can be found in [5, Definition 1][2].

When writing a category with families, we will just write $\mathbf{C}$ rather than the full tuple and let the rest be implicit. We also introduce the following notation.

Notation 3.4. Let $\mathbf{C}$ be a category with families, let $\Gamma, \Delta \in \mathbf{C}_0$, $\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma)$, $A \in \mathrm{Ty}(\Gamma)$, and $a \in \mathrm{Tm}(\Gamma, A)$. We will then use the notation

$$A[\gamma] := \mathrm{Ty}(\gamma)(A), \qquad\qquad a[\gamma] := \mathrm{Tm}(\gamma, A)(a).$$

In this new notation, the functor in Definition 3.3 becomes for each $(\Gamma, A) \in \int_{\mathbf{C}}(\mathrm{Ty})_0$

$$\begin{aligned} S_{\Gamma,A} \colon \mathbf{C}^{\mathrm{op}} &\to \mathbf{Set} \\ \Delta &\mapsto \sum_{\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta,\Gamma)} \mathrm{Tm}(\Delta, A[\gamma]) \\ \delta &\mapsto ((\gamma, a) \mapsto (\gamma \circ \delta, a[\gamma]). \end{aligned}$$

There is technically the possibility of ambiguity with this notation if either $\mathrm{Tm}(\Gamma, A) \cap \mathrm{Tm}(\Gamma, B) \neq \emptyset$ for some distinct $A, B \in \mathrm{Ty}(\Gamma)$ or $\mathrm{Ty}(\Gamma) \cap \mathrm{Tm}(\Gamma, A) \neq \emptyset$ for some $A \in \mathrm{Ty}(\Gamma)$, since for some $a \in \mathrm{Tm}(\Gamma, A) \cap \mathrm{Tm}(\Gamma, B)$, $a[\gamma]$ could mean either $\mathrm{Tm}(\gamma, A)(a)$ or $\mathrm{Tm}(\gamma, B)(a)$, and for some $a \in \mathrm{Ty}(\Gamma) \cap \mathrm{Tm}(\Gamma, A)$, $a[\gamma]$ could mean either $\mathrm{Ty}(\gamma)(a)$ or $\mathrm{Tm}(\gamma, A)(a)$. However, as long as we work in the abstract, we know of at most one valid interpretation, and thus the ambiguity becomes irrelevant.[3]

Each part of a category with families can be interpreted as a part of type theory:

- The objects of $\mathbf{C}$ are contexts.
- 1 is the empty context.
- For $\Gamma \in \mathbf{C}_0$, the elements of $\mathrm{Ty}(\Gamma)$ are the types in the context Γ.
- For $\Gamma \in \mathbf{C}_0$ and $A \in \mathrm{Ty}(\Gamma)$, the elements of $\mathrm{Tm}(\Gamma, A)$ are the terms of type A in the context Γ.
- For $\Gamma \in \mathbf{C}_0$ and $A \in \mathrm{Ty}(\Gamma)$, $\Gamma \cdot A$ corresponds to the context $\Gamma, x : A$.
- For $\Gamma, \Delta \in \mathbf{C}_0$, $\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma)$, $A \in \mathrm{Ty}(\Gamma)$, and $a \in \mathrm{Tm}(\Gamma, A)$, $A[\gamma] \in \mathrm{Ty}(\Delta)$ are $a[\gamma] \in \mathrm{Tm}(\Delta, A[\gamma])$ correspond to substitutions.

The only part we have not explained yet is the natural isomorphism, but we will postpone its explanation until we have some more results about it.

Definition 3.5. Let $\mathbf{C}$ be a category with families, and let $(\Gamma, A) \in \int_{\mathbf{C}}(\mathrm{Ty})_0$. We then define

$$(p_{\Gamma,A}, q_{\Gamma,A}) = (\Phi_{\Gamma,A})_{\Gamma \cdot A}(\mathrm{Id}_{\Gamma \cdot A}).$$

[1] Technically, if $\mathbf{C}$ is large, we cannot put it in a tuple, but in that case we instead define it as a disjoint union.

[2] They made an error in their definition; can you spot it?

[3] This similar to how for an arbitrary function $f \colon A \to B$, $f(\emptyset)$ could be either the image of $\emptyset$ under f or f applied to $\emptyset$, if $\emptyset \in A$, but if $\emptyset \in A$ is not known, the first interpretation is surely correct, as the latter interpretation is not known to be defined.

Note that in the above definition, we have $p_{\Gamma,A} \in \mathrm{Hom}_{\mathbf{C}}(\Gamma \cdot A, \Gamma)$ and $q_{\Gamma,A} \in \mathrm{Tm}(\Gamma \cdot A, A[p_{\Gamma,A}])$. This typing makes the interpretation of $p_{\Gamma,A}$ and $q_{\Gamma,A}$ clear, namely $p_{\Gamma,A}$ corresponds to weakning doing no substitution, and $q_{\Gamma,A}$ is the variable of type A introduced in the context comprehension.

Definition 3.6. Let $\mathbf{C}$ be a category with families, $(\Gamma, A) \in \int_{\mathbf{C}}(\mathrm{Ty})_0$, $\Delta \in \mathbf{C}_0$, $\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma)$, and $a \in \mathrm{Tm}(\Delta, A[\gamma])$. We then define

$$\langle \gamma, a \rangle = (\Phi_{\Gamma,A}^{-1})_{\Delta}(\gamma, a) \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma \cdot A)$$

and call it the comprehension of γ and a.

If we let $x : A$ be the variable introduced by the context comprehension $\Gamma \cdot A$, we can interpret $\langle \gamma, a \rangle$ as the substitution that substitutes a for x and acts as γ everywhere else. Note that since $\Phi_{\Gamma,A}$ is an isomorphism, this implies that a substitution $\delta \colon \Delta \to \Gamma \cdot A$ is characterized by its action on x and its action on everything else, which in a CwF is the following result.

Proposition 3.7. *Let $\mathbf{C}$ be a category with families, $(\Gamma, A) \in \int_{\mathbf{C}}(\mathrm{Ty})_0$, $\Delta \in \mathbf{C}_0$, $\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma)$, and $a \in \mathrm{Tm}(\Delta, A[\gamma])$. Then $\langle \gamma, a \rangle$ is the uniquely determined $\delta \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma \cdot a)$ such that*

$$p_{\Gamma,A} \circ \delta = \gamma, \qquad\qquad q_{\Gamma,A}[\delta] = a.$$

Proof. Let

$$\langle \gamma, a \rangle = (\Phi_{\Gamma,A}^{-1})_{\Delta}(\gamma, a).$$

It then holds that

$$\begin{aligned}
(p_{\Gamma,A} \circ \langle \gamma, a \rangle, q_{\Gamma,A}[\langle \gamma, a \rangle]) &= S_{\Gamma,A}(\langle \gamma, a \rangle)(p_{\Gamma,A}, q_{\Gamma,A}) \\
&= (S_{\Gamma,A}(\langle \gamma, a \rangle) \circ (\Phi_{\Gamma,A})_{\Gamma \cdot A})(\mathrm{Id}_{\Gamma \cdot A}) \\
&= ((\Phi_{\Gamma,A})_{\Delta} \circ \mathrm{Hom}_{\mathbf{C}}(\langle \gamma, a \rangle, \Gamma \cdot A))(\mathrm{Id}_{\Gamma \cdot A}) \\
&= (\Phi_{\Gamma,A})_{\Delta}(\langle \gamma, a \rangle) \\
&= (\gamma, a),
\end{aligned}$$

implying the equalities. Conversely, if $\delta \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma \cdot A)$ satifies the equalities, then

$$\begin{aligned}
(\gamma, a) &= (p_{\Gamma,A} \circ \delta, q_{\Gamma,A}[\delta]) \\
&= S_{\Gamma,A}(\delta)(p_{\Gamma,A}, q_{\Gamma,A}) \\
&= (S_{\Gamma,A}(\delta) \circ (\Phi_{\Gamma,A})_{\Gamma \cdot A})(\mathrm{Id}_{\Gamma \cdot A}) \\
&= ((\Phi_{\Gamma,A})_{\Delta} \circ \mathrm{Hom}_{\mathbf{C}}(\delta, \Gamma \cdot A))(\mathrm{Id}_{\Gamma \cdot A}) \\
&= (\Phi_{\Gamma,A})_{\Delta}(\delta),
\end{aligned}$$

implying that $\delta = \langle \gamma, a \rangle$. □

Is with the notation in Notation 3.4, $\langle \gamma, a \rangle$ is possibly ambiguous, as a could be a term of several distinct types, but by the same reasoning, it will never be relevant whilst working in the abstract.

The above characterization allows us to work with comprehension in two different ways as demonstrated in the following example.

Example 3.8. Let $\mathbf{C}$ be a category with families, and let $(\Gamma, A) \in \int_{\mathbf{C}}(\mathrm{Ty})_0$. We now wish to calculate $\langle p_{\Gamma,A}, q_{\Gamma,A} \rangle$. It is by Proposition 3.7 the unique $\delta \in \mathrm{Hom}_{\mathbf{C}}(\Gamma \cdot A, \Gamma \cdot A)$ such that

$$p_{\Gamma,A} \circ \delta = p_{\Gamma,A}, \qquad q_{\Gamma,A}[\delta] = q_{\Gamma,A},$$

but clearly

$$p_{\Gamma,A} \circ \mathrm{Id}_{\Gamma \cdot A} = p_{\Gamma,A}, \qquad q_{\Gamma,A}[\mathrm{Id}_{\Gamma \cdot A}] = q_{\Gamma,A},$$

and thus $\langle p_{\Gamma,A}, q_{\Gamma,A} \rangle = \mathrm{Id}_{\Gamma \cdot A}$.

Alternatively, we can note that by the proof of Proposition 3.7

$$\langle p_{\Gamma,A}, q_{\Gamma,A} \rangle = (\Phi^{-1}_{\Gamma,A})_{\Gamma \cdot A}(p_{\Gamma,A}, q_{\Gamma,A}) = (\Phi^{-1}_{\Gamma,A})_{\Gamma \cdot A}((\Phi_{\Gamma,A})_{\Gamma \cdot A}(\mathrm{Id}_{\Gamma \cdot A})) = \mathrm{Id}_{\Gamma \cdot A}\,. \tag{3.1}$$

Comprehension is well-behaved under precomposition.

Lemma 3.9. *Let* $\mathbf{C}$ *be a category with families, and let* $\Gamma, \Delta, Lambda \in \mathbf{C}_0$*,* $\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma)$*,* $\delta \in \mathrm{Hom}_{\mathbf{C}}(\Lambda, \Delta)$*,* $A \in \mathrm{Ty}(\Gamma)$*, and* $a \in \mathrm{Tm}(\Gamma, A)$*. It then holds that*

$$\langle \gamma, a \rangle \circ \delta = \langle \gamma \circ \delta, a[\delta] \rangle.$$

Proof. It holds that

$$p_{\Gamma,A} \circ \langle \gamma, a \rangle \circ \delta = \gamma \circ \delta,$$

and

$$q_{\gamma,A}[\langle \gamma, a \rangle \circ \delta] = q_{\gamma,A}[\langle \gamma, a \rangle][\delta] = a[\delta],$$

and thus by Proposition 3.7, it holds that

$$\langle \gamma, a \rangle \circ \delta = \langle \gamma \circ \delta, a[\delta] \rangle. \qquad \square$$

Using the comprehension we can extend context comprehension to a functor.

Proposition 3.10. *Let* $\mathbf{C}$ *be a category with families. Then* $- \cdot - \colon \int_{\mathbf{C}}(\mathrm{Ty})_0 \to \mathbf{C}_0$ *can be extended to a functor* $- \cdot - \colon \int_{\mathbf{C}}(\mathrm{Ty}) \to \mathbf{C}$.

Proof. Define for each $(\Gamma, A), (\Delta, B) \in \int_{\mathbf{C}}(\mathrm{Ty})_0$ and $(\gamma, A) \in \mathrm{Hom}_{\int_{\mathbf{C}}(\mathrm{Ty})}((\Delta, B), (\Gamma, A))$ (i.e. $A[\gamma] = B$)

$$\gamma \cdot A = \langle \gamma \circ p_{\Delta,B}, q_{\Delta,B} \rangle.$$

To verify that $\langle \gamma \circ p_{\Delta,B}, q_{\Delta,B} \rangle$ is sensible, note that $\gamma \circ p_{\Delta,B} \in \mathrm{Hom}_{\mathbf{C}}(\Delta \cdot B, \Gamma)$, and thus we need that $q_{\Delta,B} \in \mathrm{Tm}(\Delta \cdot B, A[\gamma \circ p_{\Delta,B}])$, but since

$$A[\gamma \circ p_{\Delta,B}] = A[\gamma][p_{\Delta,B}] = B[p_{\Delta,B}],$$

this is the case, implying that the expression is sensible. It holds for each $(\Gamma, A), (\Delta, B), (\Lambda, C) \in \int_{\mathbf{C}}(\mathrm{Ty})_0$, $(\gamma, A) \in \mathrm{Hom}_{\int_{\mathbf{C}}(\mathrm{Ty})}((\Delta, B), (\Gamma, A))$, and $(\delta, B) \in \mathrm{Hom}_{\int_{\mathbf{C}}(\mathrm{Ty})}((\Lambda, C), (\Delta, B))$ that

$$\begin{aligned} p_{\Gamma,A} \circ (\gamma \cdot A) \circ (\delta \cdot B) &= p_{\Gamma,A} \circ \langle \gamma \circ p_{\Delta,B}, q_{\Delta,B} \rangle \circ \langle \delta \circ p_{\Lambda,C}, q_{\Lambda,C} \rangle \\ &= \gamma \circ p_{\Delta,B} \circ \langle \delta \circ p_{\Lambda,C}, q_{\Lambda,C} \rangle \\ &= \gamma \circ \delta \circ p_{\Lambda,C} \end{aligned}$$

and that

$$\begin{aligned} q_{\Gamma,A}[(\gamma \cdot A) \circ (\delta \cdot B)] &= q_{\Gamma,A}[\gamma \cdot A][\delta \cdot B] \\ &= q_{\Gamma,A}[\langle \gamma \circ p_{\Delta,B}, q_{\Delta,B} \rangle][\langle \delta \circ p_{\Lambda,C}, q_{\Lambda,C} \rangle] \\ &= q_{\Delta,B}[\langle \delta \circ p_{\Lambda,C}, q_{\Lambda,C} \rangle] \\ &= q_{\Lambda,C}, \end{aligned}$$

and thus by the uniqueness of Proposition 3.7, it holds that

$$(\gamma \cdot A) \circ (\delta \cdot B) = \langle \gamma \circ \delta \circ p_{\Lambda,C}, q_{\Lambda,C} \rangle = (\gamma \circ \delta) \cdot A.$$

Finally, Example 3.8 implies that

$$\mathrm{Id}_\Gamma \cdot A = \langle p_{\Gamma,A}, q_{\Gamma,A} \rangle = \mathrm{Id}_{\Gamma \cdot A},$$

and thus $- \cdot -$ is a functor. ☐

Question for the reader: What is the type theoretic interpretation of $\gamma \cdot A \in \mathrm{Hom}_{\mathbf{C}}(\Delta \cdot A[\gamma], \Gamma \cdot A)$?

As we have now seen, a category with families corresponds to type theory with simultaneous substitution, and we will now see how to add type formers to this in correspondence with the type formers in type theory. We will only present the type former for Π-types; other type formers can be found in [5].

Definition 3.11. Let $\mathbf{C}$ be a category with families. A Π-structure on $\mathbf{C}$ consists of for each $\Gamma \in \mathbf{C}_0$, $A \in \mathrm{Ty}(\Gamma)$, and $B \in \mathrm{Ty}(\Gamma \cdot A)$ a type $\Pi_\Gamma(A, B) \in \mathrm{Ty}(\Gamma)$ and functions

$$\begin{aligned} \lambda_{\Gamma,A,B} &\colon \mathrm{Tm}(\Gamma \cdot A, B) \to \mathrm{Tm}(\Gamma, \Pi_\Gamma(A, B)) \\ \mathrm{ap}_{\Gamma,A,B} &\colon \mathrm{Tm}(\Gamma, \Pi_\Gamma(A, B)) \to \prod_{a \in \mathrm{Tm}(\Gamma, A)} \mathrm{Tm}(\Gamma, B[\langle \mathrm{Id}_\gamma, a \rangle]) \end{aligned}$$

such that for all $a \in \mathrm{Tm}(\Gamma, A)$, $b \in \mathrm{Tm}(\Gamma \cdot A, B)$, $f \in \mathrm{Tm}(\Gamma, \Pi_\Gamma(A, B))$, and $\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma)$ it holds that

$$\begin{aligned} \lambda_{\Gamma,A,B}(b)[\gamma] &= \lambda_{\Delta,A[\gamma],B[\gamma]}(b[\gamma \cdot A[\gamma]]), \\ \mathrm{ap}_{\Gamma,A,B}(f)(a)[\gamma] &= \mathrm{ap}_{\Delta,A[\gamma],B[\gamma]}(f[\gamma])(a[\gamma]), \\ \mathrm{ap}_{\Gamma,A,B}(\lambda_{\Gamma,A,B}(b))(a) &= b[\langle \mathrm{Id}_\Gamma, a \rangle], \\ \lambda_{\Gamma,A,B}(\mathrm{ap}_{\Gamma \cdot A, A[p_{\Gamma,A}], B[p_{\Gamma,A}]}(f[p_{\Gamma,A}])(q_{\Gamma,A})) &= f, \\ \Pi_\Gamma(A, B)[\gamma] &= \Pi_\Gamma(A[\gamma], B[\langle \gamma \circ p_{\Gamma,A}, q_{\Gamma,A} \rangle]). \end{aligned}$$

The various parts of this definition can be understood in terms of type theory as follows:

- The existence of $\Pi_\Gamma(A, B)$ is the formation rule.
- $\lambda_{\Gamma,A,B}$ is the introduction rule (λ-expression).
- $\mathrm{ap}_{\Gamma,A,B}$ is the elimination rule (application).
- The first equation states that introduction commutes with substitution.
- The second equation states that application commutes with substitution.
- The third equation is the computation rule (β-reduction).
- The fourth equation is the uniqueness rule (η-expansion).
- The fifth equation states that type formation commutes with substitution.

3.3 Morphisms of CwFs

As is so often the case, we can define morphisms between CwFs in order to form a category. There is however a slight hitch in that this can be done in several different ways. The question is to what extend a morphism must preserve the structure, i.e either up to equality or up to isomorphism. [5] presents the choices of either strict morphisms which preserve everything exactly and pseudo morphisms which preserve everything up to isomorphism with some coherence conditions, whilst [3] uses weak morphisms which preserve some things exactly and some things up to isomorphism. For our purposes weak morphisms are the more appropriate choice and will thus spend no more time on the others. The interested student can look at [5, Definition 4 and Definition 23] for more information on strict and psuedo morphisms of CwFs.

Definition 3.12. Let $(\mathbf{C}, 1_\mathbf{C}, \mathrm{Ty}_\mathbf{C}, \mathrm{Tm}_\mathbf{C}, - \cdot_\mathbf{C} -, \Phi^\mathbf{C}_{-,-})$ and $(\mathbf{D}, 1_\mathbf{D}, \mathrm{Ty}_\mathbf{D}, \mathrm{Tm}_\mathbf{D}, - \cdot_\mathbf{D} -, \Phi^\mathbf{D}_{-,-})$ be categories with families. A weak CwF morphism from the former to the latter is a tuple (F, σ, τ), where $F\colon \mathbf{C} \to \mathbf{D}$ is a functor, and $\sigma\colon \mathrm{Ty}_\mathbf{C} \to \mathrm{Ty}_\mathbf{D} \circ F^{\mathrm{op}}$ and $\tau\colon \mathrm{Tm}_\mathbf{C} \to \mathrm{Tm}_\mathbf{D} \circ \int_F(\sigma)^{\mathrm{op}}$ are natural transformation, such that $F(1_\mathbf{C})$ is a terminal object, and for all $(\Gamma, A) \in \int_\mathbf{C}(\mathrm{Ty})_0$ it holds that $\langle F(p_{\Gamma,A}), \tau_{(\Gamma\cdot A, A[p_{\Gamma,A}])}(q_{\Gamma,A})\rangle\colon F(\Gamma \cdot A) \to F(\Gamma) \cdot \sigma_\Gamma(A)$ is an isomorphism. We will denote the inverse as $\nu_{\Gamma,A}\colon F(\Gamma) \cdot \sigma_\Gamma(A) \to F(\Gamma \cdot A)$.

Note that when F^{op} means the functor from $\mathbf{C}^{\mathrm{op}}$ to $\mathbf{D}^{\mathrm{op}}$ which acts like F, and that this is necessary when composing but otherwise not. Let us verify that the above definition is well-typed:

- $\mathbf{C}$ and $\mathbf{D}$ are categories, and thus F being a functor makes sense.
- $\mathrm{Ty}_\mathbf{C}$ is a functor from $\mathbf{C}^{\mathrm{op}}$ to $\mathbf{Set}$, and $\mathrm{Ty}_\mathbf{D} \circ F^{\mathrm{op}}$ is a functor from $\mathbf{C}^{\mathrm{op}}$ through $\mathbf{D}^{\mathrm{op}}$ to $\mathbf{Set}$, and thus σ being a natural transformation makes sense.
- $\mathrm{Tm}_\mathbf{C}$ is a functor from $\int_\mathbf{C}(\mathrm{Ty}_\mathbf{C})^{\mathrm{op}}$ to $\mathbf{Set}$, and $\mathrm{Tm}_\mathbf{D} \circ \int_F(\sigma)^{\mathrm{op}}$ is a functor from $\int_\mathbf{C}(\mathrm{Ty}_\mathbf{C})^{\mathrm{op}}$ through $\int_\mathbf{D}(\mathrm{Ty}_\mathbf{D})^{\mathrm{op}}$ to $\mathbf{Set}$, and thus σ being a natural transformation makes sense.
- We have $p_{\Gamma,A}\colon \Gamma \cdot A \to \Gamma$, and thus $F(p_{\Gamma,A})\colon F(\Gamma \cdot A) \to F(\Gamma)$. We also have

$$q_{\Gamma,A} \in \mathrm{Tm}_\mathbf{C}(\Gamma \cdot A, A[p_{\Gamma,A}]) = \mathrm{Tm}_\mathbf{C}(\Gamma \cdot A, \mathrm{Ty}_\mathbf{C}(p_{\Gamma,A})(A))$$

and thus

$$\begin{aligned}\tau_{(\Gamma\cdot A,A[p_{\Gamma,A}])}(q_{\Gamma,A}) \in \left(\mathrm{Tm}_{\mathbf{D}} \circ \int_F (\sigma)^{\mathrm{op}}\right)(\Gamma\cdot A, \mathrm{Ty}_{\mathbf{C}}(p_{\Gamma,A})(A))\\ = \mathrm{Tm}_{\mathbf{D}}(F(\Gamma\cdot A), (\sigma_{\Gamma\cdot A}\circ \mathrm{Ty}_{\mathbf{C}}(p_{\Gamma,A}))(A))\\ = \mathrm{Tm}_{\mathbf{D}}(F(\Gamma\cdot A), ((\mathrm{Ty}_{\mathbf{D}}\circ F^{\mathrm{op}})(p_{\Gamma,A})\circ\sigma_\Gamma)(A))\\ = \mathrm{Tm}_{\mathbf{D}}(F(\Gamma\cdot A), \mathrm{Ty}_{\mathbf{D}}(F(p_{\Gamma,A}))(\sigma_\Gamma(A)))\\ = \mathrm{Tm}_{\mathbf{D}}(F(\Gamma\cdot A), \sigma_\Gamma(A)[F(p_{\Gamma,A})]).\end{aligned}$$

Combining these gives that $\langle F(p_{\Gamma,A}), \tau_{(\Gamma\cdot A,A[p_{\Gamma,A}])}(q_{\Gamma,A})\rangle\colon F(\Gamma\cdot A)\to F(\Gamma)\cdot\sigma_\Gamma(A)$ as promised.

Notation 3.13. Let $\mathbf{C}$ and $\mathbf{D}$ be categories with families, and let $(F,\sigma,\tau)\colon \mathbf{C}\to\mathbf{D}$ be a weak CwF morphism. We will then for all $\Gamma\in\mathbf{C}_0$ let $F_\Gamma = \sigma_\Gamma$, and for all $(\Gamma, A)\in \int_{\mathbf{C}}(\mathrm{Ty})_0$ let $F_{\Gamma,A} = \tau_{(\Gamma,A)}$. We will also refer to (F,σ,τ) simply as F.

Lemma 3.14. *Let $\mathbf{C}$ and $\mathbf{D}$ be categories with families, and let $F\colon\mathbf{C}\to\mathbf{D}$ be a weak CwF morphism. It then holds for all $\Delta,\Gamma\in\mathbf{C}_0$, $\gamma\in\mathrm{Hom}_{\mathbf{C}}(\Delta,\Gamma)$, $A\in\mathrm{Ty}(\Gamma)$, $a\in\mathrm{Tm}(\Gamma,A)$, and $b\in\mathrm{Tm}(\Delta,A[\gamma])$ that*

(i) $F_\Delta(A[\gamma]) = F_\Gamma(A)[F(\gamma)]$.

(ii) $F_{\Delta,A[\gamma]}(a[\gamma]) = F_{\Gamma,A}(a)[F(\gamma)]$.

(iii) $F(p_{\Gamma,A}) = p_{F(\Gamma),F_\Gamma(A)}\circ\langle F(p_{\Gamma,A}), F_{\Gamma\cdot A,A[p_{\Gamma,A}]}(q_{\Gamma,A})\rangle$.

(iv) $F_{\Gamma\cdot A,A[p_{\Gamma,A}]}(q_{\Gamma,A}) = q_{F(\Gamma),F_\Gamma(A)}[\langle F(p_{\Gamma,A}), F_{\Gamma\cdot A,A[p_{\Gamma,A}]}(q_{\Gamma,A})\rangle]$.

(v) $F(\langle\gamma,b\rangle) = \nu_{\Gamma,A}\circ\langle F(\gamma), F_{\Delta,A[\gamma]}(b)\rangle$.

Proof. (i): It holds that

$$\begin{aligned}F_\Delta(A[\gamma]) &= F_\Delta(\mathrm{Ty}_{\mathbf{C}}(\gamma)(A))\\ &= (F_\Delta\circ\mathrm{Ty}_{\mathbf{C}}(\gamma))(A)\\ &= ((\mathrm{Ty}_{\mathbf{D}}\circ F^{\mathrm{op}})(\gamma)\circ F_\Gamma)(A)\\ &= \mathrm{Ty}_{\mathbf{D}}(F(\gamma))(F_\Gamma(A))\\ &= F_\Gamma(A)[F(\gamma)].\end{aligned}$$

(ii): It holds that

$$\begin{aligned}F_{\Delta,A[\gamma]}(a[\gamma]) &= F_{\Delta,A[\gamma]}(\mathrm{Tm}_{\mathbf{C}}(\gamma,A)(a))\\ &= (F_{\Delta,A[\gamma]}\circ\mathrm{Tm}_{\mathbf{C}}(\gamma,A))(a)\\ &= \left(\left(\mathrm{Tm}_{\mathbf{D}}\circ\int_F (F_{-})^{\mathrm{op}}\right)(\gamma,A)\circ F_{\Gamma,A}\right)(a)\\ &= \mathrm{Tm}_{\mathbf{D}}(F(\gamma),F_\Gamma(A))(F_{\Gamma,A}(a))\\ &= F_{\Gamma,A}(a)[F(\gamma)].\end{aligned}$$

(iii): This follows directly from the definition of comprehension.
(iv): This follows directly from the definition of comprehension.
(v): It holds by Lemma 3.9 and (ii) that

$$\begin{aligned}\langle F(p_{\Gamma,A}), F_{\Gamma\cdot A, A[p_{\Gamma,A}]}(q_{\Gamma,A})\rangle \circ F(\langle\gamma, b\rangle) &= \langle F(p_{\Gamma,A}) \circ F(\langle\gamma, b\rangle), F_{\Gamma\cdot A, A[p_{\Gamma,A}]}(q_{\Gamma,A})[F(\langle\gamma, b\rangle)]\rangle \\ &= \langle F(p_{\Gamma,A} \circ \langle\gamma, b\rangle), F_{\Delta, A[p_{\Gamma,A}][\langle\gamma,b\rangle]}(q_{\Gamma,A}[\langle\gamma, b\rangle])\rangle \\ &= \langle F(\gamma), F_{\Delta, A[\gamma]}(b)\rangle,\end{aligned}$$

and thus

$$F(\langle\gamma, b\rangle) = \nu_{\Gamma,A} \circ \langle F(\gamma), F_{\Delta,A[\gamma]}(b)\rangle. \qquad \square$$

We see now that weak CwF morphisms preserve substitution exactly, and preserve projections and comprehension up to a canonical isomorphism.

3.4 Giraud CwFs

We will now introduce one general construction of CwFs, which by [3] is called Giraud CwFs.

Definition 3.15. Let $\mathbf{C}$ be a small finitely complete category

- Let

$$\begin{aligned}\mathrm{Ty}\colon \mathbf{C}^{\mathrm{op}} &\to \mathbf{Set} \\ \Gamma &\mapsto \{(u, v) \in \mathbf{C}_1^2 : \mathrm{Dom}_{\mathbf{C}}(u) = \Gamma, \mathrm{Cod}_{\mathbf{C}}(u) = \mathrm{Cod}_{\mathbf{C}}(v)\} \\ \gamma &\mapsto ((u, v) \mapsto (u \circ \gamma, v)),\end{aligned}$$

  i.e. for $\Gamma \in \mathbf{C}_0$, let $\mathrm{Ty}(\Gamma)$ be the set of all diagrams of the form

$$\begin{array}{ccc} & & E \\ & & \big\downarrow{\scriptstyle v} \\ \Gamma & \xrightarrow[u]{} & U \end{array},$$

  and for $\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma)$, $\mathrm{Ty}(\gamma)$ takes the above diagram to

$$\begin{array}{ccc} & & E \\ & & \big\downarrow{\scriptstyle v} \\ \Gamma & \xrightarrow[u\circ\gamma]{} & U \end{array}.$$

- Let

$$\begin{aligned}\mathrm{Tm}\colon \int_{\mathbf{C}}(\mathrm{Ty}) &\to \mathbf{Set} \\ (\Gamma, (u, v)) &\mapsto \{a \in \mathrm{Hom}_{\mathbf{C}}(\Gamma, \mathrm{Dom}_{\mathbf{C}}(v)) : u = v \circ a\} \\ (\gamma, (u, v)) &\mapsto (a \mapsto a \circ \gamma),\end{aligned}$$

  i.e. for $(\Gamma, (u, v)) \in \int_{\mathbf{C}}(\mathrm{Ty})_0$ let $\mathrm{Tm}(\Gamma, (u, v))$ be the set of commutative diagrams of the form

$$\begin{array}{ccc} & & E \\ & \overset{a}{\nearrow} & \downarrow v \\ \Gamma & \xrightarrow[u]{} & U \end{array},$$

and for $\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma)$ and $(u, v) \in \mathrm{Ty}(\Gamma)$, $\mathrm{Tm}(\gamma, (u, v))$ takes the above diagram to

$$\begin{array}{ccc} & & E \\ & \overset{a \circ \gamma}{\nearrow} & \downarrow v \\ \Gamma & \xrightarrow[u \circ \gamma]{} & U \end{array}.$$

- For $(\Gamma, (u, v)) \in \int_{\mathbf{C}} (\mathrm{Ty})_0$, let $\Gamma \cdot (u, v)$ be the pullback of (u, v), i.e.

$$\begin{array}{ccc} \Gamma \cdot (u, v) & \xrightarrow{q} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle v} \\ \Gamma & \xrightarrow[u]{} & U \end{array}$$

  is a pullback.

- For $\Delta \in \mathbf{C}_0$ and $(\Gamma, (u, v)) \in \int_{\mathbf{C}} (\mathrm{Ty})_0$, let

$$\begin{aligned} (\Phi_{\Gamma,(u,v)})_\Delta \colon \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma \cdot (u, v)) &\to S_{\Gamma,(u,v)}(\Delta) \\ \gamma &\mapsto (p \circ \gamma, q \circ \gamma), \end{aligned}$$

  where p and q are as in the above pullback diagram.

Note that both smallness and finite completeness are necessary in order to do the construction, as

- If $\mathbf{C}$ were not locally small, we could not define a presheaf on $\mathbf{C}$.
- If $\mathbf{C}$ had a proper class of objects, then for any $\Gamma \in \mathbf{C}_0$ the class of types would be large, as $(!_\Gamma, !_E)$ is a type of Γ for any $E \in \mathbf{C}_0$.
- We need a terminal object in order to satisfy the definition of a CwF, and we use pullbacks, and thus $\mathbf{C}$ must be finitely complete.

Proposition 3.16. *Let* $\mathbf{C}$ *be a small finitely complete category. Then* $\mathcal{G}\mathbf{C}$ *is a well-defined category with families.*

Proof. Functoriality of Ty and Tm is clear, and thus we need only verify that $\Phi_{\Gamma,(u,v)}$ is a natural isomorphism for all $(\Gamma,(u,v)) \in \int_{\mathbf{C}}(\mathrm{Ty})_0$. Let p and q be as in the definition, and let $\delta \in \mathrm{Hom}_{\mathbf{C}}(\Delta', \Delta)$. Note first that for $\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma \cdot A)$, we have $p \circ \gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma)$ and

$$v \circ (q \circ \gamma) = (v \circ q) \circ \gamma = (u \circ p) \circ \gamma = u \circ (p \circ \gamma), \tag{3.2}$$

so $q \circ \gamma \in \mathrm{Tm}(\Delta, \mathrm{Ty}(p \circ \gamma)(u,v))$, and thus $(p \circ \gamma, q \circ \gamma) \in S_{\Gamma,(u,v)}(\Delta)$, implying that $(\Phi_{\Gamma,(u,v)})_\Delta$ is well-defined. It holds for any $\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma \cdot (u,v))$ that

$$\begin{aligned}
((\Phi_{\Gamma,(u,v)})_{\Delta'} \circ \mathrm{Hom}_{\mathbf{C}}(\delta, \Gamma \cdot (u,v))(\gamma) &= (\Phi_{\Gamma,(u,v)})_{\Delta'}(\delta \circ \gamma) \\
&= (p \circ \delta \circ \gamma, q \circ \delta \circ \gamma) \\
&= (p \circ \gamma \circ \delta, q \circ \gamma \circ \delta) \\
&= (p \circ \gamma \circ \delta, \mathrm{Tm}(\delta, (u \circ \gamma, v))(q \circ \gamma)) \\
&= (p \circ \gamma \circ \delta, \mathrm{Tm}(\delta, \mathrm{Ty}(\gamma)(u,v))(q \circ \gamma)) \\
&= S_{\Gamma,(u,v)}(\delta)(p \circ \gamma, q \circ \gamma) \\
&= (S_{\Gamma,(u,v)}(\delta) \circ (\Phi_{\Gamma,(u,v)})_\Delta)(\gamma),
\end{aligned}$$

and thus $\Phi_{\Gamma,(u,v)}$ is a natural transformation.

For a $\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma)$, it holds that

$$\mathrm{Tm}(\Delta, \mathrm{Ty}(\gamma)(u,v)) = \mathrm{Tm}(\Delta, (u \circ \gamma, v)) = \{a \in \mathrm{Hom}_{\mathbf{C}}(\Gamma, E) : u \circ \gamma = v \circ a\},$$

and thus $S_{\Gamma,(u,v)}(\Delta)$ is the set of all pairs $(\gamma, a) \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma) \times \mathrm{Hom}_{\mathbf{C}}(\Delta, E)$ such that the square

$$\begin{array}{ccc}
\Delta & \xrightarrow{a} & E \\
{\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle v} \\
\Gamma & \xrightarrow[u]{} & U
\end{array}$$

commutes. Since (p,q) is a pullback, for any such pair there exists a unique $\rho \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \gamma \cdot (u,v))$ such that $p \circ \rho = \gamma$ and $q \circ \rho = \delta$, i.e. $(\Phi_{\Gamma,(u,v)})_\Delta(\rho) = (\gamma, a)$. This proves that $\Phi_{\Gamma,(u,v)}$ is a natural isomorphism, and thus $\mathcal{G}\mathbf{C}$ is a category with families. ☐

Corollary 3.17. *Let* $\mathbf{C}$ *be a small finitely complete category. Then for any* $(\Gamma,(u,v)) \in \int_{\mathbf{C}}(\mathrm{Ty})_0$

$$\begin{array}{ccc}
\Gamma \cdot (u,v) & \xrightarrow{q_{\Gamma,(u,v)}} & E \\
{\scriptstyle p_{\Gamma,(u,v)}}\downarrow & & \downarrow{\scriptstyle v} \\
\Gamma & \xrightarrow[u]{} & U
\end{array}$$

is a pullback square, and for any $\Delta \in \mathbf{C}_0$*,* $\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma)$*, and* $a \in \mathrm{Hom}_{\mathbf{C}}(\Delta, E)$ *such that* $u \circ \gamma = v \circ a$*,* $\langle \gamma, a \rangle \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma \cdot (u,v))$ *is the mediating morphism of* γ *and* a *in the pullback.*

Proof. Let p and q be as in Definition 3.15. It then holds that

$$(p_{\Gamma,(u,v)}, q_{\Gamma,(u,v)}) = (\Phi_{\Gamma,(u,v)})_{\Gamma\cdot(u,v)}(\mathrm{Id}_{\Gamma\cdot(u,v)}) = (p \circ \mathrm{Id}_{\Gamma\cdot(u,v)}, q \circ \circ \mathrm{Id}_{\Gamma\cdot(u,v)}) = (p, q),$$

and thus since p and q form a pullback square, so do $p_{\Gamma,(u,v)}$ and $q_{\Gamma,(u,v)}$. Let $\rho \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma \cdot (u, v))$ be the mediating morphism of γ and a in the pullback. It then holds that

$$(\Phi_{\Gamma,(u,v)})_{\Delta}(\rho) = (p \circ \rho, q \circ \rho) = (\gamma, a),$$

and thus by the construction of $\langle\gamma, a\rangle$ in the proof of Proposition 3.7, we have that $\langle\gamma, a\rangle = \rho$. □

Definition 3.18. Let $\mathbf{C}$ be a category with families. Then $\mathbf{C}$ is said to be democratic if for every $\Gamma \in \mathbf{C}_0$ there exists $\overline{\Gamma} \in \mathrm{Ty}(1)$ such that $1 \cdot \overline{\Gamma} \cong \Gamma$.

Lemma 3.19. *Let $\mathbf{C}$ be a small finitely complete category. Then $\mathcal{G}\mathbf{C}$ is democratic.*

Proof. Let $\overline{\Gamma} = (\mathrm{Id}_1, !_\Gamma)$, where $!_\Gamma$ is the unique element of $\mathrm{Hom}_{\mathbf{C}}(\Gamma, 1)$. This gives us the pullback square

$$\begin{array}{ccc} 1 \cdot \overline{\Gamma} & \xrightarrow{q_{1,\overline{\Gamma}}} & \Gamma \\ {\scriptstyle p_{1,\overline{\Gamma}}}\downarrow & & \downarrow{\scriptstyle !_\Gamma} \\ 1 & \xrightarrow[\mathrm{Id}_1]{} & 1 \end{array}$$

and since the pullback of an isomorphism is an iomorphism, $q_{1,\overline{\Gamma}} \in \mathrm{Hom}_{\mathbf{C}}(1 \cdot \overline{\Gamma}, \Gamma)$ is an isomorphism, implying that $1 \cdot \overline{\Gamma} \cong \Gamma$, and thus $\mathcal{G}\mathbf{C}$ is democratic. □

Proposition 3.20. *Let $\mathbf{C}$ and $\mathbf{D}$ be small finitely complete categories, and let $F \colon \mathbf{C} \to \mathbf{D}$ be a finitely continuous functor. Then by defining*

$$F_\Gamma(u, v) = (F(u), F(v)), \qquad\qquad F_{\Gamma,(u,v)}(a) = F(a)$$

for $\Gamma \in \mathbf{C}_0$, $(u, v) \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$, and $a \in \mathrm{Tm}_{\mathbf{C}}(\Gamma, (u, v))$, we get a weak CwF morphism $\mathcal{G}F = (F, F_{-}, F_{-,-}) \colon \mathcal{G}\mathbf{C} \to \mathcal{G}\mathbf{D}$.

Proof. It is clear for $\Gamma \in \mathbf{C}_0$ and $(u, v) \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$ that $(F(u), F(v)) \in \mathrm{Ty}_{\mathbf{C}}(F(\Gamma))$, and thus $F_\Gamma \in \mathrm{Hom}_{\mathbf{Set}}(\mathrm{Ty}_{\mathbf{C}}(\Gamma), \mathrm{Ty}_{\mathbf{D}}(F(\Gamma)))$ is well-typed. It holds for all $\Gamma, \Delta \in \mathbf{C}_0$, $\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma)$, and $(u, v) \in \mathrm{Ty}(\Gamma)$ that

$$\begin{aligned} (F_\Delta \circ \mathrm{Ty}_{\mathbf{C}}(\gamma))(u, v) &= F_\Delta(u \circ \gamma, v) \\ &= (F(u \circ \gamma), F(v)) \\ &= (F(u) \circ F(\gamma), F(v)) \\ &= (\mathrm{Ty}_{\mathbf{D}} \circ F^{\mathrm{op}})(\gamma)(F(u), F(v)) \\ &= ((\mathrm{Ty}_{\mathbf{D}} \circ F^{\mathrm{op}})(\gamma) \circ F_\Gamma)(u, v), \end{aligned}$$

and thus $F_{-} \colon \mathrm{Ty}_{\mathbf{C}} \to \mathrm{Ty}_{\mathbf{D}} \circ F$ is a well-defined natural transformation.

It is clear for $\Gamma \in \mathbf{C}_0$, $(u, v) \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$ and $a \in \mathrm{Tm}_{\mathbf{C}}(\Gamma, (u, v))$ that $F(a) \in \mathrm{Tm}_{\mathbf{C}}(F(\Gamma), F_\Gamma(u, v))$, and thus $F_{\Gamma,(u,v)} \in \mathrm{Hom}_{\mathbf{Set}}(\mathrm{Tm}_{\mathbf{C}}(\Gamma, (u, v)) \to \mathrm{Tm}_{\mathbf{D}}(F(\Gamma); F_\Gamma(u, v)))$ is well-typed. It holds for all $\Gamma, \Delta \in \mathbf{C}_0$, $\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma)$, $(u, v) \in \mathrm{Ty}(\Gamma)$, and $a \in \mathrm{Tm}(\Gamma, (u, v))$ that

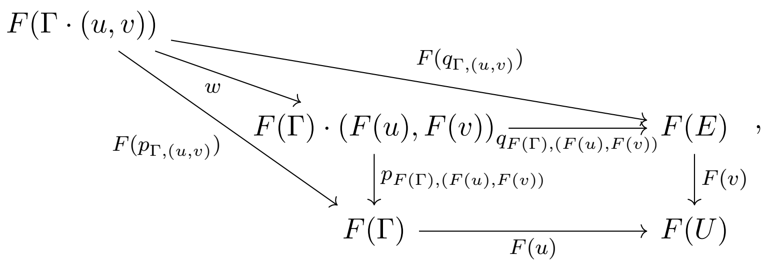
$$\begin{aligned}
(F_{\Delta,(u\circ\gamma,v)} \circ \mathrm{Tm}_{\mathbf{C}}(\gamma, (u, v)))(a) &= F_{\Delta,(u\circ\gamma,v)}(a \circ \gamma) \\
&= F(a \circ \gamma) \\
&= F(a) \circ F(\gamma) \\
&= \mathrm{Tm}_{\mathbf{D}}(F(\gamma), F_\Gamma(u, v))(F(a)) \\
&= \left(\mathrm{Tm}_{\mathbf{D}} \circ \int_F (F_{-})^{\mathrm{op}}\right)(\gamma, (u, v))(F(a)) \\
&= \left(\left(\mathrm{Tm}_{\mathbf{D}} \circ \int_F (F_{-})^{\mathrm{op}}\right)(\gamma, (u, v)) \circ F_{\Gamma,(u,v)}\right)(a),
\end{aligned}$$

and thus $F_{-,-} \colon \mathrm{Tm}_{\mathbf{C}} \to \mathrm{Tm}_{\mathbf{D}} \circ \int_F (F_{-})^{\mathrm{op}}$ is a well-defined natural transformation.

Let $\Gamma \in \mathbf{C}_0$ and $(u, v) \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$, and consider

$$\langle F(p_{\Gamma,(u,v)}), F_{\Gamma\cdot(u,v),(u\circ p_{\Gamma,(u,v)},v)}(q_{\Gamma,(u,v)})\rangle \in \mathrm{Hom}_{\mathbf{D}}(F(\Gamma \cdot (u, v)), F(\Gamma) \cdot F_\Gamma(u, v)).$$

If we call it w, Corollary 3.17 gives us the commutative diagram



where the inner square is a pullback and w is the mediating morphism. However since F is finitely continuous, the outer square is also a pullback, and thus w is an isomorphism. □

Corollary 3.21. *$\mathcal{G}$ defines a functor from the category of finitely complete categories and finitely continuous functors to the category of (democratic) categories with families and weak CwF morphisms.*

Proof. Since the definitions on Proposition 3.20 are clearly functorial, $\mathcal{G}$ becomes a functor. □

3.5 Presheaf CwFs

Another example of categories with families is any presheaf category, i.e. if $\mathbf{C}$ is a small category, we will define a CwF structure on $\widehat{\mathbf{C}}$. In order to do this properly, we need to introduce the concept of a Grothendieck universe.

Definition 3.22. A Grothendieck universe is a set $\mathcal{U}$ such that

(i) For all $u \in \mathcal{U}$ and for all $v \in u$ it holds that $v \in \mathcal{U}$.

(ii) For all $u \in \mathcal{U}$ it holds that $\mathcal{P}(u) \in \mathcal{U}$.

(iii) For all $I \in \mathcal{U}$ and for all $f\colon I \to \mathcal{U}$ it holds that $\bigcup_{i\in I} f(i) \in \mathcal{U}$.

To understand Grothendieck universes along with their motivation, imagine the following 'game': You are given a collection of sets, and you must now, using only these sets, attempt to construct a set that was not given to you. A Grothendieck universe is a collection of sets, such that this task is impossible. In other words, if the only sets that existed were the elements of a Grothendieck universe, this would be consistent with ZFC, possibly minus the axiom of infinity.

There are two immediate examples of Grothendieck universes: the empty set and the set of hereditarily finite sets (a set is hereditarily finite if it is finite and all its elements are hereditarily finite). It turns out that these two examples are the only examples that ZFC guarantees the existence of. The (informal) argument goes as follows: If a Grothendieck universe contains any hereditarily finite set, it must contain them all (this requires a proof, which we will not give here), and thus any Grothendieck universe but the above two, must contain an infinite set. This however implies that its elements satisfy *all* the axioms of ZFC, and thus ZFC cannot imply the existence of any set not an element of the Grothendieck universe, including the universe. On the other hand, it not known that ZFC excludes the existence of additional Grothendieck universes, and it is therefore often taken as an axiom when using Grothendieck universes that at least one uncountable Grothendieck universe exists if not infinitely many. We will however for now work with an arbitrary Grothendieck universe and accept that if it is trivial, the theory becomes quite basic (though in Chapter 4, we will not uncountable universes).

The interested reader may find more information on Grothendieck universes at[4][5], but here we will just state the following closure results:

Proposition 3.23. *Let $\mathcal{U}$ be a Grothendieck universe. It then holds that*

(i) If $u \in \mathcal{U}$, and $v \subseteq u$, then $u \in \mathcal{U}$.

(ii) If $u, v \in \mathcal{U}$, then $\{u, v\} \in \mathcal{U}$.

(iii) If $I \in \mathcal{U}$, and $f\colon I \to \mathcal{U}$, then $\bigcap_{u\in I} f(u) \in \mathcal{U}$.

(iv) If $I \in \mathcal{U}$, and $f\colon I \to \mathcal{U}$, then $\sum_{u\in I} f(I) \in \mathcal{U}$.

(v) If $I \in \mathcal{U}$, and $f\colon I \to \mathcal{U}$, then $\prod_{u\in I} f(u) \in \mathcal{U}$.

We will use the notation $\mathbf{Set}|_{\mathcal{U}}$ for the full subcategory of **Set** with objects in $\mathcal{U}$. Using this, we will impose a CwF structure in presheaf categories.

[4] https://ncatlab.org/nlab/show/Grothendieck+universe

[5] https://en.wikipedia.org/wiki/Grothendieck_universe

Lemma 3.24. *Let* $\mathbf{C}$ *be a small category, and let* $\mathcal{U}$ *be a Grothendieck universe. Then*

$$\begin{aligned}\mathrm{Ty}\colon \widehat{\mathbf{C}}^{\mathrm{op}} &\to \mathbf{Set}\\ \Gamma &\mapsto \mathrm{Hom}_{\mathbf{Cat}}\left(\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}}\right)\\ \gamma &\mapsto \mathrm{Hom}_{\mathbf{Cat}}\left(\int_{\mathrm{Id}_{\mathbf{C}}}(\gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}}\right)\end{aligned}$$

is a presheaf on $\widehat{\mathbf{C}}$.

Proof. Since both $\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}$ and $\mathbf{Set}|_{\mathcal{U}}$ are small, $\mathrm{Hom}_{\mathbf{Cat}}(\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})$ is a set for all $\Gamma \in \widehat{\mathbf{C}}_0$ (this is why we cannot use $\mathbf{Set}$). Since functoriality is clear, the proof is complete. □

Lemma 3.25. *Let* $\mathbf{C}$ *be a small category, let* $\mathcal{U}$ *be a Grothendieck universe, and let* Ty *be as in Lemma 3.24. Define for* $(\Gamma, A) \in \int_{\widehat{\mathbf{C}}}(\mathrm{Ty})_0$ *the set* $\mathrm{Tm}(\Gamma, A)$ *of all dependent functions* $a \in \prod_{(X,x)\in\int_{\mathbf{C}}(\Gamma)_0} A(X, x)$*, where the first argument is written as a subscript, such that for all* $X, Y \in \mathbf{C}_0$*,* $f \in \mathrm{Hom}_{\mathbf{C}}(Y, X)$*, and* $x \in \Gamma(X)$ *it holds that*

$$(A(f, x) \circ a_X)(x) = (a_Y \circ \Gamma(f))(x).$$

Define also for $(\Gamma, A) \in \int_{\widehat{\mathbf{C}}}(\mathrm{Ty})_0$*,* $\Delta \in \widehat{\mathbf{C}}_0$*,* $\gamma \in \mathrm{Hom}_{\widehat{\mathbf{C}}}(\Delta, \Gamma)$*,* $a \in \mathrm{Tm}(\Gamma, A)$*, and* $X \in \mathbf{C}_0$

$$\mathrm{Tm}(\gamma, A)(a)_X = a_X \circ \gamma_X.$$

Then Tm *is a presheaf on* $\int_{\widehat{\mathbf{C}}}(\mathrm{Ty})$.

Note that the first equation is almost the equation for a natural transformation, but because x is also an argument for A, a cannot actually be realized as a natural transformation.

Proof. It holds for all $(\Gamma, A) \in \int_{\widehat{\mathbf{C}}}(\mathrm{Ty})_0$, $\Delta \in \widehat{\mathbf{C}}$, $\gamma \in \mathrm{Hom}_{\widehat{\mathbf{C}}}(\Delta, \Gamma)$, $a \in \mathrm{Tm}(\Gamma, A)$, and $X \in \mathbf{C}_0$ that

$$\mathrm{Tm}(\gamma, A)(a)_X = a_X \circ \gamma_X \in \prod_{x\in\Delta(X)} A(X, \gamma_X(x)) = \prod_{x\in\Delta(X)} A[\gamma](X, x),$$

and furthermore for all $Y \in \mathbf{C}_0$, $f \in \mathrm{Hom}_{\mathbf{C}}(Y, X)$, and $x \in \Delta(X)$ that

$$\begin{aligned}(A(f, x) \circ \mathrm{Tm}(\gamma, A)(a)_X)(x) &= (A(f, x) \circ a_X \circ \gamma_X)(x)\\ &= (a_Y \circ \Gamma(f) \circ \gamma_X)(x)\\ &= (a_Y \circ \gamma_Y \circ \Delta(f))(x)\\ &= (\mathrm{Tm}(\gamma, A)(a)_Y \circ \Delta(f))(x),\end{aligned}$$

and thus $\mathrm{Tm}(\gamma, A)(a) \in \mathrm{Tm}(\Delta, A[\gamma])$. Since the action on morphisms is clearly functorial, the proof is complete. □

Lemma 3.26. *Let* $\mathbf{C}$ *be a small category, let* $\mathcal{U}$ *be a Grothendieck universe, and let* Ty *be as in Lemma 3.24. Then for all* $(\Gamma, A) \in \int_{\widehat{\mathbf{C}}}(\mathrm{Ty})_0$

$$\begin{aligned} \Gamma \cdot A \colon \mathbf{C}^{\mathrm{op}} &\to \mathbf{Set} \\ X &\mapsto \sum_{x \in \Gamma(X)} A(X, x) \\ f &\mapsto ((x, r) \mapsto (\Gamma(f)(x), A(f, x)(r))). \end{aligned}$$

is a presheaf on $\mathbf{C}$.

Proof. It holds for all $X, Y \in \mathbf{C}_0$, $f \colon Y \to X$, and $(x, r) \in (\Gamma \cdot A)(X)$ that $\Gamma(f)(x) \in \Gamma(Y)$ and $A(f, x)(r) \in A(Y, \Gamma(f)(x))$, and thus $(\Gamma \cdot A)(f)(x, r) \in (\Gamma \cdot A)(Y)$. It holds for all $X, Y, Z \in \mathbf{C}_0$, $f \colon Y \to X$, $g \colon Z \to Y$, and $(x, r) \in (\Gamma \cdot A)(X)$ that

$$\begin{aligned} (\Gamma \cdot A)(f \circ g)(x, r) &= (\Gamma(f \circ g)(x), A(f \circ g, x)(r)) \\ &= ((\Gamma(g) \circ \Gamma(f))(x), (A(g, \Gamma(f)(x)) \circ A(f, x))(r)) \\ &= (\Gamma \cdot A)(g)(\Gamma(f)(x), A(f, x)(r)) \\ &= ((\Gamma \cdot A)(g) \circ (\Gamma \cdot A)(f))(x, r), \end{aligned}$$

and that

$$\begin{aligned} (\Gamma \cdot A)(\mathrm{Id}_X)(x, r) &= (\Gamma(\mathrm{Id}_X)(x), A(\mathrm{Id}_X, x)(r)) \\ &= (\mathrm{Id}_{\Gamma(X)}(x), \mathrm{Id}_{A(X,x)}(r)) \\ &= (x, r), \end{aligned}$$

and thus $\Gamma \cdot A$ is a fucntor. $\square$

Lemma 3.27. *Let* $\mathbf{C}$ *be a small category, let* $\mathcal{U}$ *be a Grothendieck universe, let* Ty *be as in Lemma 3.24, let* Tm *be as in Lemma 3.25, and let* $- \cdot -$ *be as in Lemma 3.26. For* $(\Gamma, A) \in \int_{\widehat{\mathbf{C}}}(\mathrm{Ty})_0$, $\Delta \in \widehat{\mathbf{C}}_0$, $\gamma \in \mathrm{Hom}_{\widehat{\mathbf{C}}}(\Delta, \Gamma)$, $a \in \mathrm{Tm}(\Delta, A[\gamma])$, *and* $X \in \mathbf{C}_0$ *let*

$$\begin{aligned} (\Phi^{-1}_{\Gamma,A})_\Delta(\gamma, a)_X \colon \Delta(X) &\to (\Gamma \cdot A)(X) \\ x &\mapsto (\gamma_X(x), a_X(x)), \end{aligned}$$

where $^{-1}$ *is purely symbolic. Then* $\Phi^{-1}_{\Gamma,A} \colon S_{\Gamma,A} \to \mathrm{Hom}_{\widehat{\mathbf{C}}}(-, \Gamma \cdot A)$ *is a natural isomorphism, where* $S_{\Gamma,A}$ *is as in Definition 3.3.*

Proof. Fix $(\Gamma, A) \in \int_{\widehat{\mathbf{C}}}(\mathrm{Ty})_0$. It holds for all $\Delta \in \widehat{\mathbf{C}}_0$, $\gamma \in \mathrm{Hom}_{\widehat{\mathbf{C}}}(\Delta, \Gamma)$, $a \in \mathrm{Tm}(\Delta, A[\gamma])$, and $X \in \mathbf{C}_0$ that $\gamma_X(x) \in \Gamma(X)$, and $a_X(x) \in A[\gamma](X, x) = A(X, \gamma_X(x))$, and thus $(\Phi^{-1}_{\Gamma,A})_\Delta(\gamma, a)_X$ is well-typed. It holds for all $\Delta \in \widehat{\mathbf{C}}_0$, $\gamma \in \mathrm{Hom}_{\widehat{\mathbf{C}}}(\Delta, \Gamma)$, $a \in \mathrm{Tm}(\Delta, A[\gamma])$,

$X, Y \in \mathbf{C}_0$, $f \in \mathrm{Hom}_{\mathbf{C}}(Y, X)$, and $x \in \Delta(X)$ that

$$\begin{aligned}((\Gamma \cdot A)(f) \circ (\Phi^{-1}_{\Gamma,A})_\Delta(\gamma, a)_X)(x) &= (\Gamma \cdot A)(f)(\gamma_X(x), a_X(x)) \\ &= ((\Gamma(f) \circ \gamma_X)(x), (A(f, \gamma_X(x)) \circ a_X)(x)) \\ &= ((\Gamma(f) \circ \gamma_X)(x), (A[\gamma](f, x) \circ a_X)(x)) \\ &= ((\gamma_Y \circ \Delta(f))(x), (a_Y \circ \Delta(f))(x)) \\ &= (\Phi^{-1}_{\Gamma,A})_\Delta(\gamma, a)_Y(\Delta(f)(x)) \\ &= ((\Phi^{-1}_{\Gamma,A})_\Delta(\gamma, a)_Y \circ \Delta(f))(x),\end{aligned}$$

and thus $(\Phi^{-1}_{\Gamma,A})_\Delta(\gamma, a)\colon \Delta \to \Gamma \cdot A$ is a natural transformation. It holds for all $\Delta', \Delta \in \widehat{\mathbf{C}}_0$, $\delta \in \mathrm{Hom}_{\widehat{\mathbf{C}}}(\Delta', \Delta)$, $\gamma \in \mathrm{Hom}_{\widehat{\mathbf{C}}}(\Delta, \Gamma)$, $a \in \mathrm{Tm}(\Delta, A[\gamma])$, and $X \in \mathbf{C}_0$ that

$$\begin{aligned}(\mathrm{Hom}_{\widehat{\mathbf{C}}}(\delta, \Gamma \cdot A) \circ (\Phi^{-1}_{\Gamma,A})_\Delta)(\gamma, a)_X &= \mathrm{Hom}_{\widehat{\mathbf{C}}}(\delta, \Gamma \cdot A)_X(x \mapsto (\gamma_X(x), a_X(x))) \\ &= (x \mapsto (\gamma_X(\delta_X(x)), a_X(\delta_X(x)))) \\ &= (x \mapsto ((\gamma \circ \delta)_X(x), a[\delta]_X(x))) \\ &= (\Phi^{-1}_{\Gamma,A})_{\Delta'}(\gamma \circ \delta, a[\gamma])_X \\ &= ((\Phi^{-1}_{\Gamma,A})_{\Delta'} \circ S_{\Gamma,A}(\delta))(\gamma, a)_X,\end{aligned}$$

and thus $\Phi^{-1}_{\Gamma,A}\colon S_{\Gamma,A} \to \mathrm{Hom}_{\widehat{\mathbf{C}}}(-, \Gamma, A)$ is a natural transformation. It is clear from the definition of $\Phi^{-1}_{\Gamma,A}$ that all of its components are injective. To see that they are also surjective, let $\Delta \in \widehat{\mathbf{C}}$ and $\eta \in \mathrm{Hom}_{\widehat{\mathbf{C}}}(\Delta, \Gamma \cdot A)$, and let for each $X \in \mathbf{C}_0$ and $x \in \Delta(X)$ $\gamma_X(x)$ and $a_X(x)$ be respectively the first and second coordinate of $\eta_X(x) \in (\Gamma \cdot A)(X) = \sum_{x \in \Gamma(X)} A(X, x)$. It then holds for all $X, Y \in \mathbf{C}_0$, $f \in \mathrm{Hom}_{\mathbf{C}}(Y, X)$, and $x \in \Delta(X)$ that

$$\begin{aligned}((\Gamma(f) \circ \gamma_X)(x), (A(f, x) \circ a_X)(x)) &= (\Gamma \cdot A)(f)(\gamma_X(x), a_X(x)) \\ &= ((\Gamma \cdot A)(f) \cdot \eta_X)(x) \\ &= (\eta_Y \circ \Delta(f))(x) \\ &= ((\gamma_Y \circ \Delta(f))(x), (a_Y \circ \Delta(f))(x)),\end{aligned}$$

and thus $\gamma\colon \Delta \to \Gamma$ is a natural transformation and $a \in \mathrm{Tm}(\Delta, A[\gamma])$ is a term, and since for all $X \in \mathbf{C}_0$ and $x \in \Delta(X)$

$$(\Phi^{-1}_{\Gamma,A})_\Delta(\gamma, a)_X(x) = (\gamma_X(x), a_X(x)) = \eta_X(x),$$

this proves that $\Phi^{-1}_{\Gamma,A}$ has surjective components and is thus a natural isomorphism. □

Theorem 3.28. *Let* $\mathbf{C}$ *be a small category, let* $\mathcal{U}$ *be a Grothendieck universe, let* Ty *be as in Lemma 3.24, let* Tm *be as in Lemma 3.25, let* $- \cdot -$ *be as in Lemma 3.26, let* $\Phi^{-1}_{-,-}$ *be as in Lemma 3.27, and let* $1_{\widehat{\mathbf{C}}}$ *be the presheaf on* $\mathbf{C}$ *that is constantly* $1 = \{\emptyset\}$*. Then* $(\widehat{\mathbf{C}}, 1_{\widehat{\mathbf{C}}}, \mathrm{Ty}, \mathrm{Tm}, - \cdot -, \Phi_{-,-})$ *is a CwF.*

Given the above use of presheafs on categories of elements, now seems an appropriate to prove the following equivalence of categories.

Theorem 3.29. *Let* $\mathbf{C}$ *be a small posetal category, let* $\mathcal{U}$ *be a class satisfying the axioms of a Grothendieck universe, and let* $\Gamma \in \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})$*. Then*

$$\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})/\Gamma \simeq \mathrm{Fun}\left(\int_{\mathbf{C}} (\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}}\right).$$

Proof. We will first define a functor $L\colon {}^{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})}\!/_{\Gamma} \to \mathrm{Fun}(\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})$. For each $\Delta \in \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})_0$ and $\gamma \in \mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})}(\Delta, \Gamma)$, we define

$$\begin{aligned} L(\gamma)\colon \int_{\mathbf{C}}(\Gamma)^{\mathrm{op}} &\to \mathbf{Set}|_{\mathcal{U}} \\ (c, x) &\mapsto \gamma_c^{-1}(\{x\}) \\ (d \le c, x) &\mapsto (r \mapsto \Delta(d \le c)(r)). \end{aligned}$$

Since for $(x, c) \in \int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}$, $\gamma_c^{-1}(\{x\}) \subseteq \Delta(c) \in \mathcal{U}$, the action on objects is well-defined by Proposition 3.23. The action on morphisms is well-defined since for any $d \le c \in \mathbf{C}_1$, $x \in \Gamma(c)$, and $r \in \gamma_c^{-1}(\{x\})$ it holds that

$$\gamma_d(L(\gamma)(d \le c, x)(r)) = \gamma_d(\Delta(d \le c)(r)) = \Gamma(d \le c)(\gamma_c(r)) = \Gamma(d \le c)(x),$$

and thus

$$L(\gamma)(d \le c, x)(r) \in \gamma_d^{-1}(\{\Gamma(d \le c)(x)\}) = L(\gamma)(d, \Gamma(d \le c)(x)).$$

Functoriality is clear. For all $\Delta, \Delta' \in \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})_0$, $\gamma' \in \mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})}(\Delta', \Gamma)$, $\gamma \in \mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})}(\Delta, \Gamma)$, $\delta \in \mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})/_{\Gamma}}(\gamma', \gamma)$, and $(c, x) \in \int_{\mathbf{C}}(\Gamma)_0$, we define

$$\begin{aligned} L(\delta)_{(c,x)}\colon L(\gamma')(c, x) &\to L(\gamma)(c, x) \\ r &\mapsto \delta_c(r). \end{aligned}$$

This is well-defined since for any $r \in L(\gamma')(c, x)$

$$\gamma_c(L(\delta)_{(c,x)}(r)) = \gamma_c(\delta_c(r)) = \gamma'_c(r) = x,$$

which implies that

$$L(\delta)_{(c,x)}(r) \in \gamma_c^{-1}(\{x\}) = L(\gamma)(c, x).$$

This is a natural transformation since for any $d \le c \in \mathbf{C}_1$, it holds that

$$\begin{aligned} (L(\gamma)(d \le c, x) \circ L(\delta)_{(c,x)})(r) &= L(\gamma)(d \le c, x)(\delta_c(r)) \\ &= \Delta(d \le c)(\delta_c(r)) \\ &= \delta_d(\Delta'(d \le c)(r)) \\ &= L(\delta)_{(d,\Gamma(d\le c)(x))}(\Delta'(d \le c)(r)) \\ &= (L(\delta)_{(d,\Gamma(d\le c)(x))} \circ L(\gamma')(d \le c, x))(r). \end{aligned}$$

Functoriality of L is clear.

We will next define a functor $R\colon \mathrm{Fun}(\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}}) \to {}^{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})}\!/_{\Gamma}$, but before that we will define a functor $\Sigma\colon \mathrm{Fun}(\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}}) \to \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})$. For $A \in \mathrm{Fun}(\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})_0$, let

$$\begin{aligned} \Sigma(A)\colon \mathbf{C}^{\mathrm{op}} &\to \mathbf{Set}|_{\mathcal{U}} \\ c &\mapsto \sum_{x\in\Gamma(c)} A(c, x) \\ d \le c &\mapsto ((x, r) \mapsto (\Gamma(d \le c)(x), A(d \le c, x)(r))). \end{aligned}$$

Note that this is just $\Gamma \cdot A$ from Lemma 3.26, and thus it is a functor, and by Proposition 3.23 it goes to $\mathbf{Set}|_{\mathcal{U}}$. For $A, B \in \mathrm{Fun}(\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})_0$, $\varphi \in \mathrm{Hom}_{\mathrm{Fun}(\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})}(A, B)$, and $c \in \mathbf{C}_0$, we define

$$\begin{aligned} \Sigma(\varphi)_c \colon \Sigma(A)(c) &\to \Sigma(B)(c) \\ (x, r) &\mapsto (x, \varphi_{(c,x)}(r)), \end{aligned}$$

which is well-defined since for $(x, r) \in \Sigma(A)(c)$, we have $\varphi_{(c,x)}(r) \in B(c, x)$, and thus $\Sigma(\varphi)_{(c,x)}(x, r) \in \Sigma(B)(c)$. it holds for all $A, B \in \mathrm{Fun}(\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})_0$, $\varphi \in \mathrm{Hom}_{\mathrm{Fun}(\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})}(A, B)$, $d \leq c \in \mathbf{C}_1$, and $(x, r) \in \Sigma(A)(c)$ that

$$\begin{aligned} (\Sigma(B)(d \leq c) \circ \Sigma(\varphi)_c)(x, r) &= \Sigma(B)(d \leq c)(x, \varphi_{(c,x)}(r)) \\ &= (\Gamma(d \leq c)(x), (B(d \leq c, x) \circ \varphi_{(c,x)})(r)) \\ &= (\Gamma(d \leq c)(x), (\varphi_{(d, \Gamma(d \leq c)(x))} \circ A(d \leq c, x))(r)) \\ &= \Sigma(\varphi)_d(\Gamma(d \leq c)(x), A(d \leq c, x)(r)) \\ &= (\Sigma(\varphi)_d \circ \Sigma(A)(d \leq c))(x, r), \end{aligned}$$

and thus $\Sigma(\varphi) \colon \Sigma(A) \to \Sigma(B)$ is a natural transformation, which implies, since Σ's action on morphisms is clearly functorial, that Σ is a functor.

For each $A \in \mathrm{Fun}(\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})$ and $c \in \mathbf{C}$, we define

$$\begin{aligned} R(A)_c \colon \Sigma(A)(c) &\to \Gamma(c) \\ (x, r) &\mapsto x. \end{aligned}$$

This is clearly a natural transformation (and is in fact $p_{\Gamma,A}$). Define for all $A, B \in \mathrm{Fun}(\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})_0$ and $\varphi \in \mathrm{Hom}_{\mathrm{Fun}(\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})}(A, B)$ that $R(\varphi) = \Sigma(\varphi)$. Since

$$(R(B) \circ R(\varphi))_c(x, r) = R(B)_c(x, \varphi(c, x)(r)) = x = R(A)_c(x, r)$$

for all for $c \in \mathbf{C}_0$ and $(x, r) \in \Sigma(A)(c)$, and thus $R(\varphi) \colon R(A) \to R(B)$, implying that R is a well-defined functor.

We will define a natural transformation $\eta' \colon \Sigma \circ L \to U_\Gamma$, where $U_\Gamma \colon {}^{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})}\!/_{\Gamma} \to \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})$ is the forgetful functor. Note first that for $\Delta \in \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})_0$, $\gamma \in \mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})}(\Delta, \Gamma)$, and $c \in \mathbf{C}_0$ it holds that

$$(\Sigma \circ L)(\gamma)(c) = \sum_{x \in \Gamma(c)} L(\gamma)(c, x) = \sum_{x \in \Gamma(c)} \gamma_c^{-1}(\{x\})$$

and

$$U_\Gamma(\gamma)(c) = \Delta(c),$$

and we may thus define

$$\begin{aligned} (\eta'_\gamma)_c \colon (\Sigma \circ L)(\gamma)(c) &\to U_\Gamma(\gamma)(c) \\ (x, r) &\mapsto r, \end{aligned}$$

which is an isomorphism with inverse $r \mapsto (\gamma_c(r), r)$. It holds for $\Delta \in \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})_0$, $\gamma \in \mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})}(\Delta, \Gamma)$, $d \leq c \in \mathbf{C}_0$, and $(x, r) \in (\Sigma \circ L)(\gamma)(c)$ that

$$\begin{aligned} ((\eta'_\gamma)_d \circ (\Sigma \circ L)(\gamma)(d \leq c))(x, r) &= (\eta'_\gamma)_d(\Gamma(d \leq c)(x), L(\gamma)(d \leq c, x)(r)) \\ &= (\eta'_\gamma)_d(\Gamma(d \leq c)(x), \Delta(d \leq c)(r)) \\ &= \Delta(d \leq c)(r) \\ &= U_\Gamma(\gamma)(d \leq c)(r) \\ &= (U_\Gamma(\gamma)(d \leq c) \circ (\eta'_\gamma)_c)(x, r), \end{aligned}$$

and thus $\eta'_\gamma\colon (\Sigma\circ L)(\gamma) \to U_\Gamma(\gamma)$ is a natural isomorphism. It holds for all $\Delta, \Delta' \in \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})_0$, $\gamma \in \mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})}(\Delta,\Gamma)$, $\gamma' \in \mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})}(\Delta',\Gamma)$, $\delta \in \mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})/\Gamma}(\gamma',\gamma)$, $c \in \mathbf{C}_0$, and $(x,r) \in (\Sigma\circ L)(\gamma)(c)$ that

$$\begin{aligned}(\eta'_\gamma \circ (\Sigma\circ L)(\delta))_c(x,r) &= (\eta'_\gamma)_c(x, L(\delta)_{(c,x)}(r))\\ &= L(\delta)_{(c,x)}(r)\\ &= \delta_c(r)\\ &= U_\Gamma(\delta)_c(r)\\ &= (U_\Gamma(\delta)\circ \eta'_{\gamma'})_c(x,r),\end{aligned}$$

and thus $\eta'\colon \Sigma\circ L \to U_\Gamma$ is a natural isomorphism.

It holds for all $\Delta \in \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})_0$, $\gamma \in \mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})}(\Delta,\Gamma)$, $c \in \mathbf{C}$, and $(x,r) \in (\Sigma\circ L)(\gamma)(c)$ that

$$\begin{aligned}(\mathrm{Id}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})/\Gamma}(\gamma)\circ\eta'_\gamma)_c(x,r) &= \mathrm{Id}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})/\Gamma}(\gamma)_c(r)\\ &= \gamma_c(r)\\ &= x\\ &= (R\circ L)(\gamma)_c(x,r),\end{aligned}$$

where the penultimate equality follows from $r \in L(\gamma)(c,x) = \gamma_c^{-1}(\{x\})$, and thus $\eta'_\gamma \in \mathrm{Hom}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})/\Gamma}((R\circ L)(\gamma), \mathrm{Id}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})/\Gamma}(\gamma))$, which implies that we may define a natural isomorphism $\eta\colon R\circ L \to \mathrm{Id}_{\mathrm{Fun}(\mathbf{C}^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})/\Gamma}$ by $\eta_\gamma = \eta'_\gamma$.

It holds for $A \in \mathrm{Fun}(\textstyle\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})_0$ and $(c,x) \in \textstyle\int_{\mathbf{C}}(\Gamma)_0$

$$\begin{aligned}(L\circ R)(A)(c,x) &= R(A)_c^{-1}(\{x\})\\ &= \left(\sum_{y\in\Gamma(c)} A(c,y) \ni (z,r) \mapsto z \in \Gamma(c)\right)^{-1}(\{x\})\\ &= \{x\}\times A(c,x),\end{aligned}$$

and thus we may define for $A \in \mathrm{Fun}(\textstyle\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})$ and $(c,x) \in \textstyle\int_{\mathbf{C}}(\Gamma)_0$ the isomorphism

$$\begin{aligned}(\varepsilon_A)_{(c,x)}\colon A(c,x) &\to (L\circ R)(A)(c,x)\\ r &\mapsto (x,r).\end{aligned}$$

It holds for $A \in \mathrm{Fun}(\textstyle\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})$, $(d\le c, x) \in \textstyle\int_{\mathbf{C}}(\Gamma)_1$, and $r \in A(c,x)$ that

$$\begin{aligned}((\varepsilon_A)_{(d,\Gamma(d\le c)(x))}\circ A(d\le c,x))(r) &= (\Gamma(d\le c)(x), A(d\le c,x)(r))\\ &= \Sigma(A)(d\le c)(x,r)\\ &= (L\circ R)(A)(d\le c,x)(x,r)\\ &= ((L\circ R)(A)(d\le c,x)\circ(\varepsilon_A)_{(c,x)})(r),\end{aligned}$$

and thus $\varepsilon_A\colon A \to (L\circ R)(A)$ is a natural isomorphism. It holds for all $A, B \in \mathrm{Fun}(\textstyle\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})_0$, $\varphi \in \mathrm{Hom}_{\mathrm{Fun}(\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})}(A,B)$, $(c,x) \in \textstyle\int_{\mathbf{C}}(\Gamma)_0$, and $r \in A(c,x)$ that

$$\begin{aligned}(\varepsilon_B\circ \mathrm{Id}_{\mathrm{Fun}(\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}},\mathbf{Set}|_{\mathcal{U}})}(\varphi))_{(c,x)}(r) &= (\varepsilon_B)_{(c,x)}(\varphi_{(c,x)}(r))\\ &= (x, \varphi_{(c,x)}(r))\\ &= R(\varphi)_c(x,r)\\ &= (L\circ R)(\varphi)_{(c,x)}(x,r)\\ &= ((L\circ R)(\varphi)\circ\varepsilon_A)_{(c,x)}(r),\end{aligned}$$

and thus $\varepsilon\colon \mathrm{Id}_{\mathrm{Fun}(\int_{\mathbf{C}}(\Gamma)^{\mathrm{op}},\mathbf{C}|_{\sqcap})} \to L \circ R$ is a natural isomorphism.

Combining all of the above we see that L and R form an equivalence of categories. $\square$

Corollary 3.30. *Let* $\mathbf{C}$ *be a small posetal category, let* $\mathcal{U}$ *be a Grothendieck universe, and consider* $\widehat{\mathbf{C}}$ *as a CwF via Theorem 3.28. It then holds for any* $\Gamma \in \widehat{\mathbf{C}}_0$ *that there exists* $\overline{\Gamma} \in \mathrm{Ty}(1)$ *such that* $\Gamma \cong 1 \cdot \overline{\Gamma}$ *if and only if* Γ *is isomorphic to an element of* $\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})_0$.

Proof. Assume there exists $\overline{\Gamma} \in \mathrm{Ty}(1)$ such that $\Gamma \cong 1 \cdot \overline{\Gamma}$. Using the notation from the proof of Theorem 3.29 with $\Gamma = 1$, it holds that

$$1 \cdot \overline{\Gamma} = \Sigma(\overline{\Gamma}) \in \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}}),$$

which proves the first implication.

Assume instead that Γ is isomorphic to an element of $\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})_0$, call it Δ. It then holds again using the notation form the proof of Theorem 3.29 with $\Gamma = 1$ that

$$\Gamma \cong \Delta \simeq (\Sigma \circ L)(!_{\Delta}) = 1 \cdot L(!_{\Delta}),$$

which proves the second implication. $\square$

Corollary 3.31. *Let* $\mathbf{C}$ *be a small posetal category, let* $\mathcal{U}$ *be a non-empty Grothendieck universe, and consider* $\widehat{\mathbf{C}}$ *as a CwF via Theorem 3.28. Then* $\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})$ *is a small full democratic sub-CwF of* $\widehat{\mathbf{C}}$.

Proof. It holds that $\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})$ is a small full subcategory of $\widehat{\mathbf{C}}$. It holds for any $\Gamma \in \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})_0$ and $A \in \mathrm{Ty}(\Gamma)$ using the notation of the proof of Theorem 3.29 that

$$\Gamma \cdot A = \Sigma(A) \in \mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}}),$$

and thus $\mathrm{Fun}(\mathbf{C}^{\mathrm{op}}, \mathbf{Set}|_{\mathcal{U}})$ is a sub-CwF. It follows from Corollary 3.30 that it is democratic $\square$

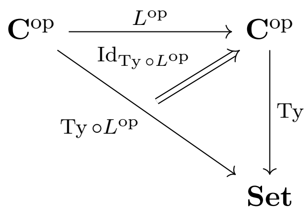
It is worth noting that whilst we never used any properties of $\mathcal{U}$ other than it being a set in the proof of Theorem 3.28, it is essential in the above corollaries that it be closed under subsets and dependent pairs as this is used in Theorem 3.29. It is possible to add both Π-types and Σ-types to presheaf CwFs, which also requires that $\mathcal{U}$ be a Grothendieck universe. Details can be found at [6, Section 4.2].

3.6 Categories with Dependent Right Adjoints

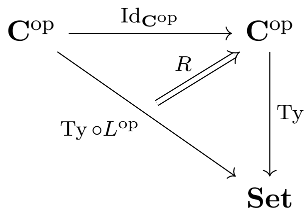
We will now see how to add modal operators to CwFs.

Definition 3.32. A category with dependent right adjoints is a tuple $(\mathbf{C}, L, R, \Psi)$, where $\mathbf{C}$ is a category with families, $L\colon \mathbf{C} \to \mathbf{C}$ is a functor, $R\colon \mathrm{Ty} \circ L^{\mathrm{op}} \to \mathrm{Ty}$ is a natural transformation, and $\Psi\colon \mathrm{Tm} \circ \int_L (\mathrm{Id}_{\mathrm{Ty} \circ L^{\mathrm{op}}})^{\mathrm{op}} \to \mathrm{Tm} \circ \int_{\mathrm{Id}_{\mathbf{C}^{\mathrm{op}}}} (R)^{\mathrm{op}}$ is a natural isomorphism.

Let us verify that Ψ is well-typed: $\int$ takes the diagram



to the functor $\int_L(\mathrm{Id}_{\mathrm{Ty}\circ L^{\mathrm{op}}})$, which goes from $\int_{\mathbf{C}}(\mathrm{Ty}\circ L^{\mathrm{op}})$ to $\int_{\mathbf{C}}(\mathrm{Ty})$, and thus $\mathrm{Tm}\circ\int_L(\mathrm{Id}_{\mathrm{Ty}\circ L^{\mathrm{op}}})^{\mathrm{op}}$ goes from $\int_{\mathbf{C}}(\mathrm{Ty}\circ L^{\mathrm{op}})^{\mathrm{op}}$ through $\int_{\mathbf{C}}(\mathrm{Ty})^{\mathrm{op}}$ to **Set**. $\int$ also takes the diagram



to the functor $\int_{\mathrm{Id}_{\mathbf{C}^{\mathrm{op}}}}(R)$, which goes from $\int_{\mathbf{C}}(\mathrm{Ty}\circ L^{\mathrm{op}})$ to $\int_{\mathbf{C}}(\mathrm{Ty})$, and thus $\mathrm{Tm}\circ\int_{\mathrm{Id}_{\mathbf{C}^{\mathrm{op}}}}(R)^{\mathrm{op}}$ goes from $\int_{\mathbf{C}}(\mathrm{Ty}\circ L^{\mathrm{op}})^{\mathrm{op}}$ through $\int_{\mathbf{C}}(\mathrm{Ty})^{\mathrm{op}}$ to **Set**. Since these two functors go from the same category to the same category, Ψ is well-typed.

If we try to unpack the two isomorphic functors, we get for all $\Gamma\in\mathbf{C}_0$ and $A\in\mathrm{Ty}(L(\Gamma))$

$$\left(\mathrm{Tm}\circ\int_L(\mathrm{Id}_{\mathrm{Ty}\circ L^{\mathrm{op}}})^{\mathrm{op}}\right)(\Gamma,A)=\mathrm{Tm}(L(\Gamma),A) \tag{3.3}$$

and

$$\left(\mathrm{Tm}\circ\int_{\mathrm{Id}_{\mathbf{C}^{\mathrm{op}}}}(R)^{\mathrm{op}}\right)(\Gamma,A)=\mathrm{Tm}(\Gamma,R_\Gamma(A)), \tag{3.4}$$

and thus $\Psi_{(\Gamma,A)}$ gives that

$$\mathrm{Tm}(L(\Gamma),A)\cong\mathrm{Tm}(\Gamma,R_\Gamma(A)).$$

Note that this looks at lot like the usual definition of adjoints, which is the reason behind the name.

With regards to interpretation in terms of type theory, L and R model respectively 🔒 and $\Box$, since L acts on contexts like 🔒 and R takes a type of $L(\Gamma)$ to a type of Γ like $\Box$. Note further the above isomorphism, which models the bijection between the terms of A in $\Gamma,$🔒 and the terms of $\Box A$ in Γ.

Theorem 3.33. *Let* $\mathbf{C}$ *be a category with families, and let* $L\dashv R$ *be adjoint endofunctors on* $\mathbf{C}$ *such that* R *can be extended to a weak CwF morphism. Then* $\mathbf{C}$ *can be made into a category with dependent right adjoints with* L *as the endofunctor.*

Proof. Let $\eta\colon \mathrm{Id}_{\mathbf{C}} \to R \circ L$ be the unit of the adjunction, and let $\varepsilon\colon L \circ R \to \mathrm{Id}_{\mathbf{C}}$ be the counit of the adjunction. We define the natural transformation $R^*\colon \mathrm{Ty} \circ L^{\mathrm{op}} \to \mathrm{Ty}$ as

$$\begin{aligned} R^*_\Gamma\colon \mathrm{Ty}(L(\Gamma)) &\to \mathrm{Ty}(\Gamma) \\ A &\mapsto R_{L(\Gamma)}(A)[\eta_\Gamma] \end{aligned}$$

for $\Gamma \in \mathbf{C}_0$. Let us verify that this is well-typed: We have $A \in \mathrm{Ty}(L(\Gamma))$, and since $R_-\colon \mathrm{Ty} \to \mathrm{Ty} \circ R^{\mathrm{op}}$, we have $R_{L(\Gamma)}\colon \mathrm{Ty}(L(\Gamma)) \to \mathrm{Ty}(R(L(\Gamma)))$, implying that $R_{L(\Gamma)}(A) \in \mathrm{Ty}(R(L(\Gamma)))$. We have $\eta\colon \mathrm{Id}_{\mathbf{C}} \to R \circ L$, and thus $\eta_\Gamma\colon \Gamma \to R(L(\Gamma))$, implying that $\mathrm{Ty}(\eta_\Gamma)\colon \mathrm{Ty}(R(L(\Gamma))) \to \mathrm{Ty}(\Gamma)$, which finally gives us that $R_{L(\Gamma)}[\eta_\Gamma] \in \mathrm{Ty}(\Gamma)$.

Let $\Gamma, \Delta \in \mathbf{C}_0$, and let $\gamma \in \mathrm{Hom}_{\mathbf{C}}(\Delta, \Gamma)$. It then holds for all $A \in \mathrm{Ty}(L(\Gamma))$ by Lemma 3.14(1) that

$$\begin{aligned} (\mathrm{Ty}(\gamma) \circ R^*_\Gamma)(A) &= \mathrm{Ty}(\gamma)(R_{L(\Gamma)}(A)[\eta_\Gamma]) \\ &= R_{L(\Gamma)}(A)[\eta_\Gamma][\gamma] \\ &= R_{L(\Gamma)}(A)[\eta_\Gamma \circ \gamma] \\ &= R_{L(\Gamma)}(A)[R(L(\gamma)) \circ \eta_\Delta] \\ &= R_{L(\Gamma)}(A)[R(L(\gamma))][\eta_\Delta] \\ &= R_{L(\Delta)}(A[L(\gamma)])[\eta_\Delta] \\ &= R^*_\Delta(A[L(\gamma)]) \\ &= (R^*_\Delta \circ \mathrm{Ty}(L(\gamma)))(A), \end{aligned}$$

and thus R^* is natural.

We define the natural transformation

$$\begin{aligned} \Psi_{(\Gamma,A)}\colon \mathrm{Tm}\left(\int_L (\mathrm{Id}_{\mathrm{Ty} \circ L^{\mathrm{op}}})(\Gamma, A)\right) &\to \mathrm{Tm}\left(\int_{\mathrm{Id}_{\mathbf{C}^{\mathrm{op}}}} (R^*)(\Gamma, A)\right) \\ a &\mapsto R_{L(\Gamma),A}(a)[\eta_\Gamma] \end{aligned}$$

for $(\Gamma, A) \in \int_{\mathbf{C}} (\mathrm{Ty} \circ L^{\mathrm{op}})_0$. Note that by equation (3.3) and (3.4), $\Psi_{(\Gamma,A)}$ actually goes from $\mathrm{Tm}(L(\Gamma), A)$ to $\mathrm{Tm}(\Gamma, R^*_\Gamma(A))$, and by the same arguments as for R^* the above definition is well-typed.

Let $(\Gamma, A), (\Delta, B) \in \int_{\mathbf{C}} (\mathrm{Ty} \circ L^{\mathrm{op}})_0$, and let $(\gamma, A) \in \mathrm{Hom}_{\int_{\mathbf{C}} (\mathrm{Ty} \circ L^{\mathrm{op}})}((\Delta, B), (\Gamma, A))$ (note that $A[\gamma] = B$). It then holds for all $a \in \mathrm{Tm}(L(\Gamma), A)$ by Lemma 3.14(2) that

$$\begin{aligned} \left(\left(\mathrm{Tm} \circ \int_{\mathrm{Id}_{\mathbf{C}^{\mathrm{op}}}} (R^*)^{\mathrm{op}}\right)(\gamma, A) \circ \Psi_{(\Gamma,A)}\right)(a) &= \mathrm{Tm}(\gamma, R_\Gamma(A))(R_{L(\Gamma),A}(a)[\eta_\Gamma]) \\ &= R_{L(\Gamma),A}(a)[\eta_\Gamma][\gamma] \\ &= R_{L(\Gamma),A}(a)[\eta_\Gamma \circ \gamma] \\ &= R_{L(\Gamma),A}(a)[R(L(\gamma)) \circ \eta_\Delta] \\ &= R_{L(\Gamma),A}(a)[R(L(\gamma))][\eta_\Delta] \\ &= R_{L(\Delta),B}(a[L(\gamma)])[\eta_\Delta] \\ &= \Psi_{(\Delta,B)}(a[L(\gamma)]) \\ &= (\Psi_{(\Delta,B)})(\mathrm{Tm}(L(\gamma), A)(a)) \\ &= \left(\Psi_{(\Delta,B)} \circ \left(\mathrm{Tm} \circ \int_L (\mathrm{Id}_{\mathrm{Ty} \circ L^{\mathrm{op}}})\right)(\gamma, A)\right)(a) \end{aligned}$$

and thus Ψ is natural.

Let $\nu_{\Gamma,A} \colon R(\Gamma) \cdot R_\Gamma(A) \to R(\Gamma \cdot A)$ be the inverse of $\langle R(p_{\Gamma,A}), R_{\Gamma\cdot A, A[p_{\Gamma,A}]}(q_{\Gamma,A})\rangle$ for each $(\Gamma, A) \in \int_{\mathbf{C}}(\mathrm{Ty})_0$. We define for each $(\Gamma, A) \in \int_{\mathbf{C}}(\mathrm{Ty} \circ L^{\mathrm{op}})_0$

$$\Omega_{(\Gamma,A)} \colon \mathrm{Tm}\left(\int_{\mathrm{Id}_{\mathbf{C}^{\mathrm{op}}}} (R^*)(\Gamma, A)\right) \to \mathrm{Tm}\left(\int_L (\mathrm{Id}_{\mathrm{Ty}\circ L^{\mathrm{op}}})(\Gamma, A)\right)$$
$$b \mapsto q_{L(\Gamma),A}[\varepsilon_{L(\Gamma)\cdot A} \circ L(\nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle)].$$

As with Ψ, equations (3.3) and (3.4) give that $\Omega_{(\Gamma,A)}$ goes from $\mathrm{Tm}(\Gamma, R^*_\Gamma(A))$ to $\mathrm{Tm}(L(\Gamma), A)$. We have that $\eta_\Gamma \colon \Gamma \to R(L(\Gamma))$ and $b \in \mathrm{Tm}(\Gamma, R^*_\Gamma(A)) = \mathrm{Tm}(\Gamma, R_{L(\Gamma)}(A)[\eta_\Gamma])$, and thus $\langle \eta_\Gamma, b\rangle \colon \Gamma \to R(L(\Gamma)) \cdot R_{L(\Gamma)}(A)$ is well-defined. Since $\nu_{L(\Gamma),A} \colon R(L(\Gamma)) \cdot R_{L(\Gamma)}(A) \to R(L(\Gamma) \cdot A)$, we have $\nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle \colon \Gamma \to R(L(\Gamma) \cdot A)$, implying that $L(\nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle) \colon L(\Gamma) \to L(R(L(\Gamma \cdot A))$. Since $\varepsilon_{L(\Gamma)\cdot A} \colon L(R(L(\Gamma) \cdot A)) \to L(\Gamma) \cdot A$, this gives that $\varepsilon_{L(\Gamma)\cdot A} \circ L(\nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle) \colon L(\Gamma) \to L(\Gamma) \cdot A$, and thus since $q_{L(\Gamma),A} \in \mathrm{Tm}(L(\Gamma) \cdot A, A[p_{L(\Gamma),A}])$, we have $q_{L(\Gamma),A}[\varepsilon_{L(\Gamma)\cdot A} \circ L(\nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle)] \in \mathrm{Tm}(L(\Gamma), A[p_{L(\Gamma),A}][\varepsilon_{L(\Gamma)\cdot A} \circ L(\nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle)])$. It holds by Lemma 3.14(3) that

$$\begin{aligned}
p_{L(\Gamma),A} \circ \varepsilon_{L(\Gamma)\cdot A} \circ L(\nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle) &= \varepsilon_{L(\Gamma)} \circ L(R(p_{L(\Gamma),A})) \circ L(\nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle) \\
&= \varepsilon_{L(\Gamma)} \circ L(R(p_{L(\Gamma),A}) \circ \nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle) \\
&= \varepsilon_{L(\Gamma)} \circ L(p_{R(L(\Gamma)),R_{L(\Gamma)}(A)} \circ \langle \eta_\Gamma, b\rangle) \\
&= \varepsilon_{L(\Gamma)} \circ L(\eta_\Gamma) \\
&= \mathrm{Id}_{L(\Gamma)},
\end{aligned}$$

and thus $q_{L(\Gamma),A}[\varepsilon_{L(\Gamma)\cdot A} \circ L(\nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle)] \in \mathrm{Tm}(L(\Gamma), A)$ as intended, implying that $\Omega_{(\Gamma,A)}$ is well-typed.

It holds for all $(\Gamma, A) \in \int_{\mathbf{C}}(\mathrm{Ty} \circ L^{\mathrm{op}})_0$, $a \in \mathrm{Tm}(L(\Gamma), A)$, and $b \in \mathrm{Tm}(\Gamma, R^*_\Gamma(a))$ by Lemma 3.9 and Lemma 3.14(4) that

$$\begin{aligned}
(\Omega_{(\Gamma,A)} \circ \Psi_{(\Gamma,A)})(a) &= \Omega_{(\Gamma,A)}(R_{L(\Gamma),A}(a)[\eta_\Gamma]) \\
&= q_{L(\Gamma),A}[\varepsilon_{L(\Gamma)\cdot A} \circ L(\nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, R_{L(\Gamma),A}(a)[\eta_\Gamma]\rangle)] \\
&= q_{L(\Gamma),A}[\varepsilon_{L(\Gamma)\cdot A} \circ L(\nu_{L(\Gamma),A} \circ \langle \mathrm{Id}_{R(L(\Gamma))}, R_{L(\Gamma),A}(a)\rangle \circ \eta_\Gamma)] \\
&= q_{L(\Gamma),A}[\varepsilon_{L(\Gamma)\cdot A} \circ L(\nu_{L(\Gamma),A} \circ \langle R(\mathrm{Id}_{L(\Gamma)}), R_{L(\Gamma),A}(a)\rangle \circ \eta_\Gamma)] \\
&= q_{L(\Gamma),A}[\varepsilon_{L(\Gamma)\cdot A} \circ L(R(\langle \mathrm{Id}_{L(\Gamma)}, a\rangle) \circ \eta_\Gamma)] \\
&= q_{L(\Gamma),A}[\varepsilon_{L(\Gamma)\cdot A} \circ L(R(\langle \mathrm{Id}_{L(\Gamma)}, a\rangle)) \circ L(\eta_\Gamma)] \\
&= q_{L(\Gamma),A}[\langle \mathrm{Id}_{L(\Gamma)}, a\rangle \circ \varepsilon_{L(\Gamma)} \circ L(\eta_\Gamma)] \\
&= q_{L(\Gamma),A}[\langle \mathrm{Id}_{L(\Gamma)}, a\rangle] \\
&= a,
\end{aligned}$$

and by Lemma 3.14(2,4), Lemma 3.9, and Examples 3.8 that

$$\begin{aligned}
(\Psi_{(\Gamma,A)} \circ \Omega_{(\Gamma,A)})(b) &= \Psi_{(\Gamma,A)}(q_{L(\Gamma),A}[\varepsilon_{L(\Gamma)\cdot A} \circ L(\nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle)]) \\
&= R_{L(\Gamma),A}(q_{L(\Gamma),A}[\varepsilon_{L(\Gamma)\cdot A} \circ L(\nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle)])[\eta_\Gamma] \\
&= R_{L(\Gamma)\cdot A, A[p_{L(\Gamma),A}]}(q_{L(\Gamma),A})[R(\varepsilon_{L(\Gamma)\cdot A} \circ L(\nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle)) \circ \eta_\Gamma] \\
&= R_{L(\Gamma)\cdot A, A[p_{L(\Gamma),A}]}(q_{L(\Gamma),A})[R(\varepsilon_{L(\Gamma)\cdot A}) \circ R(L(\nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle)) \circ \eta_\Gamma] \\
&= R_{L(\Gamma)\cdot A, A[p_{L(\Gamma),A}]}(q_{L(\Gamma),A})[R(\varepsilon_{L(\Gamma)\cdot A}) \circ \eta_{R(L(\Gamma)\cdot A)} \circ \nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle] \\
&= R_{L(\Gamma)\cdot A, A[p_{L(\Gamma),A}]}(q_{L(\Gamma),A})[\nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle] \\
&= q_{R(L(\Gamma)),R_{L(\Gamma)}(A)}[\langle R(p_{L(\Gamma),A}), R_{L(\Gamma)\cdot A, A[p_{L(\Gamma),A}]}(q_{L(\Gamma),A})\rangle][\nu_{L(\Gamma),A} \circ \langle \eta_\Gamma, b\rangle] \\
&= q_{R(L(\Gamma)),R_{L(\Gamma)}(A)}[\langle R(p_{L(\Gamma),A}) \circ \nu_{L(\Gamma),A}, R_{L(\Gamma)\cdot A, A[p_{L(\Gamma),A}]}(q_{L(\Gamma),A}) \circ \nu_{L(\Gamma),A}\rangle \circ \langle \eta_\Gamma, b\rangle] \\
&= q_{R(L(\Gamma)),R_{L(\Gamma)}(A)}[\langle p_{R(L(\Gamma)),R_{L(\Gamma)}(A)}, q_{R(L(\Gamma)),R_{L(\Gamma)}(A)}\rangle \circ \langle \eta_\Gamma, b\rangle] \\
&= q_{R(L(\Gamma)),R_{L(\Gamma)}(A)}[\langle \eta_\Gamma, b\rangle] \\
&= b,
\end{aligned}$$

and thus Ψ is a linear isomoprhism. □

Corollary 3.34. *Let* $\mathbf{C}$ *be a small finitely complete category, and let* $L \dashv R$ *be adjoint endofunctors on* $\mathbf{C}$*. Then* $\mathbf{C}$ *can be given the structure of a category with dependent right adjoint with* L *as the endofunctor.*

Proof. Since R is a right adjoint, it is continuous, and thus by Proposition 3.20 it extends to a weak CwF morphism, implying by Theorem 3.33 that $\mathbf{C}$ can be given the structure of a CwDRA. □

Since we can most certainly find a small finitely complete category and an endoajunction on it, this tells us that CwDRAs indeed do exist and are in fact quite plentiful. The simplest example we can construct with the above corollary uses the adjunction $\mathrm{Id}_\mathbf{C} \dashv \mathrm{Id}_\mathbf{C}$, but this just makes the dependent right adjoint the identity, which we could probably have figured out was an option by just looking at the definition. Instead we will consider a slightly more interesting example:

Example 3.35. Let $\mathbf{C}$ be a small finitely complete category with exponentials, let $\Lambda \in \mathbf{C}$, and consider the adjunction $\Lambda \times - \dashv -^\Lambda$. Since this notation does not work well with subscripts, we will be sometimes using L and R instead. First, the extension of $-^\Lambda$ to a weak CwF morphism using Proposition 3.20: We define for $\Gamma \in \mathbf{C}_0$ that

$$\begin{aligned}
R_\Gamma \colon \mathrm{Ty}(\Gamma) &\to \mathrm{Ty}\left(\Gamma^\Lambda\right) \\
(u, v) &\mapsto \left(u^\Lambda, v^\Lambda\right)
\end{aligned}$$

and for $(\Gamma, (u, v)) \in \int_\mathbf{C}(\mathrm{Ty})_0$ that

$$\begin{aligned}
R_{\Gamma,(u,v)} \colon \mathrm{Tm}(\Gamma, (u, v)) &\to \mathrm{Tm}\left(\Gamma^\Lambda, \left(u^\Lambda, v^\Lambda\right)\right) \\
a &\mapsto a^\Lambda.
\end{aligned}$$

For $\Gamma \in \mathbf{C}_0$, the component at Γ of the unit of the adjunction is the exponential transpose $\eta_\Gamma = \widetilde{\mathrm{Id}_{\Lambda\times\Gamma}} \colon \Gamma \to (\Lambda \times \Gamma)^\Lambda$, i.e. we have the diagram

$$
\begin{array}{ccc}
\Lambda \times (\Lambda \times \Gamma)^\Lambda & \xrightarrow{\epsilon} & \Lambda \times \Gamma \\
{\scriptstyle \mathrm{Id}_\Lambda \times \widetilde{\mathrm{Id}_{\Lambda\times\Gamma}}} \uparrow & \nearrow {\scriptstyle \mathrm{Id}_{\Lambda\times\Gamma}} & \\
\Lambda \times \Gamma & &
\end{array}
,
$$

where ϵ is the evaluation morphism (and the component at Γ of the counit). Note the special property of $\widetilde{\mathrm{Id}_{\Lambda\times\Gamma}}$ that $u^\Lambda \circ \widetilde{\mathrm{Id}_{\Lambda\times\Gamma}} = \widetilde{u}$ for all $u \colon \Lambda \times \Gamma \to U$. This leads to the definitions

$$
\begin{aligned}
R^*_\Gamma \colon \ \mathrm{Ty}(\Lambda \times \Gamma) &\to \mathrm{Ty}(\Gamma) \\
(u, v) &\mapsto R_{L(\Gamma)}(u, v)[\eta_\Gamma] \\
&= \left(u^\Lambda, v^\Lambda\right)[\eta_\Gamma] \\
&= \left(u^\Lambda \circ \eta_\Gamma, v^\Lambda\right) \\
&= \left(\widetilde{u}, v^\Lambda\right)
\end{aligned}
$$

for $\Gamma \in \mathbf{C}_0$ and

$$
\begin{aligned}
\Psi_{(\Gamma,(u,v))} \colon \ \mathrm{Tm}(\Lambda \times \Gamma, (u, v)) &\to \mathrm{Tm}\left(\Gamma, \left(\widetilde{u}, v^\Lambda\right)\right) \\
a &\mapsto R_{L(\Gamma),(u,v)}(a)[\eta_\Gamma] \\
&= a^\Lambda[\eta_\Gamma] \\
&= a^\Lambda \circ \eta_\Gamma \\
&= \widetilde{a}
\end{aligned}
$$

for $(\Gamma, (u, v)) \in \int_{\mathbf{C}} (\mathrm{Ty} \circ L^{\mathrm{op}})_0$.

Chapter 4

The Topos of Trees

We will in this chapter focus our attention to the topos of trees. If we let ω denote the posetal category constructed from the natural numbers (excluding 0) with the usual total order, we can then define the topos of trees as $\mathbf{S} = \widehat{\omega} = \mathbf{Set}^{\omega^{\text{op}}}$. An object $\Gamma \in \mathbf{S}_0$ can be drawn as the diagram

$$\Gamma(1) \xleftarrow{\Gamma(1\leq 2)} \Gamma(2) \xleftarrow{\Gamma(2\leq 3)} \Gamma(3) \xleftarrow{\Gamma(3\leq 4)} \cdots ,$$

and for $\Delta \in \mathbf{S}_0$ a morphism $\gamma \in \mathrm{Hom}_{\mathbf{S}}(\Delta, \Gamma)$ can be drawn as the commutative diagram

$$\begin{array}{ccccccc}
\Delta(1) & \xleftarrow{\Delta(1\leq 2)} & \Delta(2) & \xleftarrow{\Delta(2\leq 3)} & \Delta(3) & \xleftarrow{\Delta(3\leq 4)} & \cdots \\
{\scriptstyle \gamma_1}\downarrow & & {\scriptstyle \gamma_2}\downarrow & & {\scriptstyle \gamma_3}\downarrow & & \\
\Gamma(1) & \xleftarrow{\Gamma(1\leq 2)} & \Gamma(2) & \xleftarrow{\Gamma(2\leq 3)} & \Gamma(3) & \xleftarrow{\Gamma(3\leq 4)} & \cdots
\end{array} .$$

Notation 4.1. Let $\Gamma \in \mathbf{S}_0$, let $i, j \in \omega_0$ with $j \leq i$, and let $x \in X(i)$. We then define

$$x|_j = X(j \leq i)(x).$$

This notation is potentially ambiguous, as x might be an element of $\Gamma(i)$ for several different i's, but the meaning should be clear from context in most situations, and if it is not, we will default back to $\Gamma(j \leq i)(x)$.

4.1 Subobjects

Since $\mathbf{S}$ is a presheaf category, it is cartesian closed (with pointwise limits), cocomplete (with pointwise colimits) and has subobject classifier. Before we describe the subobject classifier, we will give a description of the subobjects in $\mathbf{S}$.

Lemma 4.2. *Let $\Gamma, \Delta \in \mathbf{S}_0$, and let $\mu\colon \Delta \to \Gamma$. Then μ is monic if and only if μ_i is monic for all $i \in \omega_0$, and in the confirming case there exist uniquely determined subsets $\Lambda(i) \subseteq \Gamma(i)$ for $i \in \omega_0$ with $\Gamma(j \leq i)(\Lambda(i)) \subseteq \Lambda(j)$ for all $(j \leq i) \in \omega_1$, such that the inclusion morphism of*

$$\Lambda(1) \xleftarrow{\Gamma(1\leq 2)} \Lambda(2) \xleftarrow{\Gamma(2\leq 3)} \Lambda(3) \xleftarrow{\Gamma(3\leq 4)} \cdots$$

into Γ *is isomorphic to* μ*.*

Proof. The first statement is known since ω^{op} has pullbacks. Let $\Lambda(i) = \mu_i(\Delta(i))$ for $i \in \omega_i$. It holds for $i, j \in \omega_i$ with $j \leq i$, $x \in \Lambda(i)$, and $y \in \mu_i^{-1}(a)$ that

$$\Gamma(j \leq i)(x) = (\Gamma(j \leq i) \circ \mu_i)(y) = (\mu_j \circ \Delta(j \leq i))(y) \in \Lambda(j),$$

and thus $\Gamma(j \leq i)(\Lambda(i)) \subseteq \Lambda(j)$. The definition of Λ gives the diagram

$$\begin{array}{ccc} & & \Gamma \\ & \overset{\mu}{\nearrow} & \Big\uparrow \iota \\ \Delta & \xrightarrow[\mu]{} & \Lambda \end{array} ,$$

where ι is the inclusion morphism, and $\mu\colon \Delta \to \Lambda$ is the corestriction of μ to Λ. Since the components of $\mu\colon \Delta \to \Lambda$ are monic (i.e. injective) by the first statement, and they are surjective by the definition of Λ, $\mu\colon \Delta \to \Lambda$ is an isomorphism, proving that $\mu\colon \Delta \to \Gamma$ and ι are isomorphic.

Let $\iota_\Lambda\colon \Lambda \to \Gamma$ and $\iota_B\colon B \to \Gamma$ be inclusion morphisms isomorphic via $f\colon \Lambda \to B$. It then holds for all $i \in \omega_0$ and $x \in \Lambda(i)$ that

$$a = (\iota_\Lambda)_i(x) = ((\iota_B)_i \circ f_i)(x) = f_i(x),$$

implying that f is the identity, and thus $\iota_\Lambda = \iota_B$, proving the uniqueness of Λ. □

From now on we will, when we speak of subobjects, we will assume they are inclusion morphisms, and we will consider appropriate subsets as subobjects via their inclusion morphism.

Proposition 4.3. *Let* $\Omega \in \mathbf{S}_0$ *be*

$$\{0,1\} \xleftarrow{\min(1,-)} \{0,1,2\} \xleftarrow{\min(2,-)} \{0,1,2,3\} \xleftarrow{\min(3,-)} \cdots ,$$

let true$\colon 1 \to \Omega$*, where* 1 *is the terminal object, be*

$$\begin{array}{ccccccc} 1 & \xleftarrow{\ \ !\ \ } & 1 & \xleftarrow{\ \ !\ \ } & 1 & \xleftarrow{\ \ !\ \ } & \cdots \\ {\scriptstyle 1}\big\downarrow & & {\scriptstyle 2}\big\downarrow & & {\scriptstyle 3}\big\downarrow & & \\ \{0,1\} & \xleftarrow{\min(1,-)} & \{0,1,2\} & \xleftarrow{\min(2,-)} & \{0,1,2,3\} & \xleftarrow{\min(3,-)} & \cdots \end{array} ,$$

and define for each subobject of $\Gamma \in \mathbf{S}_0$, Λ, *the morphism* $\chi_\Lambda \colon \Gamma \to \Omega$ *with components defined as*

$$(\chi_\Lambda)_i \colon \Gamma(i) \to \Omega(i)$$
$$x \mapsto \begin{cases} \max\{j \in \Omega(i) \setminus \{0\} : x|_j \in \Lambda(j)\} & \exists j \in \Omega(i) \setminus \{0\}(x|_j \in \Lambda(j)) \\ 0 & \forall j \in \Omega(i) \setminus \{0\}(x|_j \notin \Lambda(j)) \end{cases}$$

for $i \in \omega_0$. *This defines a subobject classifier in* $\mathbf{S}$.

Proof. It is clear that true is a natural transformation. Let $i, j \in \omega_0$ with $j \leq i$, and let $x \in \Gamma(i)$. In order to prove that χ_Λ is a natural transformation, we must prove that $(\chi_\Lambda)_j(x|_j) = (\chi_\Lambda)_i(x)|_j$. Assume first that $(\chi_\Lambda)_i(x) = 0$. Then $(\chi_\Lambda)_i(x)|_j = 0$, and $x|_k \notin \Lambda(k)$ for all $k \in \Omega(k) \setminus \{0\}$, and thus $(\chi_\Lambda)_j(x|_j) = 0$, implying the equality. Assume next that $(\chi_\Lambda)_i(x) \in \Omega(j) \setminus \{0\}$. This implies that $(\chi_\Lambda)_i(x)|_j = (\chi_\Lambda)_i(x)$, $x|_{(\chi_\Lambda)_i(x)} \in \Lambda((\chi_\Lambda)_i(x))$, and $x|_k \notin \Lambda(k)$ for all $k \in \Omega(i) \setminus \Omega((\chi_\Lambda)_i(x))$, and thus $(\chi_\Lambda)_j(x|_j) = (\chi_\Lambda)_i(x)$, implying the equality. Assume last that $(\chi_\Lambda)_i(x) \in \Omega(i) \setminus \Omega(j)$. This implies that $(\chi_\Lambda)_i(x)|_j = j$, and $x|_j \in \Lambda(j)$ by the naturality of the inclusion morphism, and thus $(\chi_\Lambda)_j(x|_j) = j$, implying the equality. This proves that χ_Λ is a natural transformation.

In order to prove that this is a subobject classifier, we must prove that the diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{!_\Lambda} & 1 \\ {\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \text{true}} \\ \Gamma & \xrightarrow[\chi_\Lambda]{} & \Omega \end{array}$$

is a pullback diagram. It holds for all $i \in \omega_0$ and $a \in \Lambda(i)$ that $(\text{true} \circ !_\Lambda)_i(a) = i$ and that $(\chi_\Lambda \circ \iota)_i(a) = (\chi_\Lambda)_i(a) = i$, and thus the above diagram commutes. Let $\Delta \in \mathbf{S}_0$, let $\gamma \colon \Delta \to \Gamma$, and assume that $\chi_\Lambda \circ \gamma = \text{true} \circ !_\Delta$. Since for any $i \in \omega_0$ and $y \in \Delta(i)$, we have $(\text{true} \circ !_\Delta)_i(y) = i$, we also have $(\chi_\Lambda \circ \gamma)_i(y) = i$, and thus $\gamma_i(y) \in \Gamma(i)$. This implies that γ factors through ι with some $\delta \colon \Delta \to \Lambda$, and since ι is monic, this δ is unique. By the uniqueness of $!_\Delta$, we also have that $!_\Lambda \circ \delta = !_\Delta$, and thus δ is the mediating morphism, implying that the diagram is indeed a pullback diagram. □

There is an important endomorphism $\triangleright \colon \Omega \to \Omega$

$$\begin{array}{ccccccc} \{0,1\} & \xleftarrow{\min(1,-)} & \{0,1,2\} & \xleftarrow{\min(2,-)} & \{0,1,2,3\} & \xleftarrow{\min(3,-)} & \cdots \\ {\scriptstyle \min(0,-)+1}\downarrow & & {\scriptstyle \min(1,-)+1}\downarrow & & {\scriptstyle \min(2,-)+1}\downarrow & & \\ \{0,1\} & \xleftarrow{\min(1,-)} & \{0,1,2\} & \xleftarrow{\min(2,-)} & \{0,1,2,3\} & \xleftarrow{\min(3,-)} & \cdots \end{array} .$$

Since Ω is a subobject classifier, there exists a natural isomorphism $\theta \colon \mathbf{Sub}_{\mathbf{Sub}} \to \mathrm{Hom}_{\mathbf{S}}(-, \Omega)$, and we may thus define the natural transformation $\triangleright = \theta^{-1} \circ \mathrm{Hom}_{\mathbf{S}}(-, \triangleright) \circ \theta \colon \mathbf{Sub}_{\mathbf{S}} \to \mathbf{Sub}_{\mathbf{S}}$ (different from $\triangleright \colon \Omega \to \Omega$), where $\mathbf{Sub}_{\mathbf{S}} \colon \mathbf{S}^{\mathrm{op}} \to \mathbf{Set}$ is the subobject functor using the representatives from Lemma 4.2. We thus have for each $\Gamma \in \mathbf{S}_0$ and $\Lambda \in \mathbf{Sub}_{\mathbf{S}}(\Gamma)$ the pullback diagram

$$\begin{array}{ccc}
\triangleright_\Gamma(\Lambda) & \xrightarrow{!_{\triangleright_\Gamma(\Lambda)}} & 1 \\
{\scriptstyle \iota}\downarrow & & \downarrow{\scriptstyle \mathrm{true}} \\
\Gamma & \xrightarrow[\triangleright\circ\chi_\Lambda]{} & \Omega
\end{array}$$

Proposition 4.4. *Let $\Gamma \in \mathbf{S}_0$, and let $\Lambda \in \mathbf{Sub}_{\mathbf{S}}(\Gamma)$. Then*

$$\Gamma(1) \xleftarrow{\Gamma(1\leq 2)} \Gamma(1 \leq 2)^{-1}(\Lambda(1)) \xleftarrow{\Gamma(2\leq 3)} \Gamma(2 \leq 3)^{-1}(\Lambda(2)) \xleftarrow{\Gamma(3\leq 4)} \cdots$$

is equal to $\triangleright_\Gamma(\Lambda)$.

Proof. Call the subobject Λ'. It is sufficient to prove that $\chi_{\Lambda'} = \triangleright \circ \chi_\Lambda$. Fix $i \in \omega_0$ and $x \in \Lambda'(i)$, and let $j = (\chi_\lambda)_i(x)$. If $j = 0$ we have $x|_1 \in \Gamma(1) = \Lambda'(1)$, and thus $(\chi_{\Lambda'})_i(x) \geq 1$. If $0 < j < i$ we have $(x|_{j+1})|_j = x|_j \in \Lambda(i)$, and thus $x|_{j+1} \in \Gamma(j \leq j+1)^{-1}(\Lambda(j)) = \Lambda'(j+1)$, and thus $(\chi_{\Lambda'})_i(x) \geq j+1$. If $j+1 < i$ and $(\chi_{\Lambda'})_i(x) > j+1$ we have $x|_{j+2} \in \Lambda'(j+2) = \Gamma(j+1 \leq j+2)^{-1}(\Lambda(j+1))$, and thus $x|_{j+1} = (x|_{j+2})|_{j+1} \in \Lambda(j+1)$, which is a contradiction, and thus $(\chi_{\Lambda'})(x) = j+1$. If $j+1 = i$, then $j+1 \leq (\chi_{\Lambda'})_i(x) \leq i$, implying that $(\chi_{\Lambda'})_i(x) = j+1$. If $j = i = 1$ we have $x \in \Gamma(1) = \Lambda'(1)$, implying that $(\chi_{\Lambda'})_i(x) = i$. If $j = i > 1$ we have $x \in \Lambda(i)$, implying $x|_{i-1} \in \Lambda(i-1)$, and thus $x \in \Gamma(i-1 \leq i)^{-1}(\Lambda(i-1)) = \Lambda'(i)$, implying that $(\chi_{\Lambda'})_i(x) = i$. Putting all of it together we see that $(\chi_\Lambda)_i(x) = \min(i-1, j)+1 = (\triangleright\circ\chi_\Lambda)_i(x)$, which completes the proof. □

4.2 The Later Modality

Of particular interest in this context is a particular endofunctor on $\mathbf{S}$, called the later modality, which is denoted ▶. It takes $\Gamma \in \mathbf{S}$ to

$$1 \xleftarrow{!_{\Gamma(1)}} \Gamma(1) \xleftarrow{\Gamma(1\leq 2)} \Gamma(2) \xleftarrow{\Gamma(2\leq 3)} \cdots\ ,$$

and for $\Delta \in \mathbf{S}$ it takes $\gamma \in \mathrm{Hom}_{\mathbf{S}}(\Delta, \Gamma)$ to

$$\begin{array}{ccccccc}
1 & \xleftarrow{!_{\Delta(1)}} & \Delta(1) & \xleftarrow{\Delta(1\leq 2)} & \Delta(2) & \xleftarrow{\Delta(2\leq 3)} & \cdots \\
{\scriptstyle !_1}\downarrow & & {\scriptstyle \gamma_1}\downarrow & & {\scriptstyle \gamma_2}\downarrow & & \\
1 & \xleftarrow{!_{\Gamma(1)}} & \Gamma(1) & \xleftarrow{\Gamma(1\leq 2)} & \Gamma(2) & \xleftarrow{\Gamma(2\leq 3)} & \cdots
\end{array}\ .$$

Proposition 4.5. *Let $\mathcal{U}$ be a non-empty Grothendieck universe, and consider $\mathbf{S}$ as a category with families via Theorem 3.28. Then $\blacktriangleright$ extends to a weak CwF morphism.*

Proof. Note first that $\blacktriangleright(1) = 1$.

For $\Gamma \in \mathbf{S}_0$ and $A \in \mathrm{Ty}(\Gamma)$, define

$$\blacktriangleright_\Gamma(A)\colon \int_\omega (\blacktriangleright(\Gamma))^{\mathrm{op}} \to \mathbf{Set}|_{\mathcal{U}}$$
$$(i,x) \mapsto \begin{cases} A(i-1,x) & i>1 \\ 1 & i=1 \end{cases}$$
$$(j \le i, x) \mapsto \begin{cases} A(j-1 \le i-1, x) & j>1 \\ !_{\blacktriangleright_\Gamma(A)(i,x)} & j=1 \end{cases},$$

which is clearly a functor. It holds for all $\Gamma, \Delta \in \mathbf{S}_0$, $\gamma \in \mathrm{Hom}_{\mathbf{S}}(\Delta, \Gamma)$, $A \in \mathrm{Ty}(\Gamma)$, $(i,x) \in \int_\omega(\blacktriangleright(\Delta))_0$, and $j \le i \in \omega_1$ if $i > 1$ that

$$\begin{aligned}(\blacktriangleright_\Delta \circ \mathrm{Ty}(\gamma))(A)(i,x) &= \mathrm{Ty}(\gamma)(A)(i-1,x) \\ &= A(i-1, \gamma_{i-1}(x)) \\ &= A(i-1, \blacktriangleright(\gamma)_i(x)) \\ &= \blacktriangleright_\Gamma(A)(i, \blacktriangleright(\gamma)_i(x)) \\ &= ((\mathrm{Ty} \circ \blacktriangleright)(\gamma) \circ \blacktriangleright_\Gamma)(A)(i,x),\end{aligned}$$

if $j > 1$ that

$$\begin{aligned}(\blacktriangleright_\Delta \circ \mathrm{Ty}(\gamma))(A)(j \le i,x) &= \mathrm{Ty}(\gamma)(A)(j-1 \le i-1,x) \\ &= A(j-1 \le i-1, \gamma_{i-1}(x)) \\ &= A(j-1 \le i-1, \blacktriangleright(\gamma)_i(x)) \\ &= \blacktriangleright_\Gamma(A)(j \le i, \blacktriangleright(\gamma)_i(x)) \\ &= ((\mathrm{Ty} \circ \blacktriangleright)(\gamma) \circ \blacktriangleright_\Gamma)(A)(j \le i,x),\end{aligned}$$

if $i = 1$ that

$$\begin{aligned}(\blacktriangleright_\Delta \circ \mathrm{Ty}(\gamma))(A)(i,x) &= 1 \\ &= \blacktriangleright_\Gamma(A)(i, \blacktriangleright(\gamma)_i(x)) \\ &= ((\mathrm{Ty} \circ \blacktriangleright)(\gamma) \circ \blacktriangleright_\Gamma)(A)(i,x),\end{aligned}$$

and if $j = 1$ that

$$\begin{aligned}(\blacktriangleright_\Delta \circ \mathrm{Ty}(\gamma))(A)(j \le i,x) &= !_{(\blacktriangleright_\Delta \circ \mathrm{Ty}(\gamma))(A)(i,x)} \\ &= !_{((\mathrm{Ty} \circ \blacktriangleright)(\gamma) \circ \blacktriangleright_\Gamma)(A)(i,x)} \\ &= !_{(\blacktriangleright_\Gamma(A)(j \le i, \blacktriangleright(\gamma)_i(x)))} \\ &= \blacktriangleright_\Gamma(A)(j \le i, \blacktriangleright(\gamma)_i(x)) \\ &= ((\mathrm{Ty} \circ \blacktriangleright)(\gamma) \circ \blacktriangleright_\Gamma)(A)(j \le i,x),\end{aligned}$$

and thus $\blacktriangleright_{\_}\colon \mathrm{Ty} \to \mathrm{Ty} \circ \blacktriangleright$ is a natural transformation.

For $\Gamma \in \mathbf{S}_0$, $A \in \mathrm{Ty}(\Gamma)$, $a \in \mathrm{Tm}(\Gamma, A)$, and $i \in \omega_0$, define

$$\begin{aligned} \blacktriangleright_{\Gamma,A}(a)_i \colon \blacktriangleright(\Gamma)(i) &\to \blacktriangleright_\Gamma(A)(i,-) \\ x &\mapsto \begin{cases} a_{i-1}(x) & i > 1 \\ 0 & i = 1 \end{cases}. \end{aligned}$$

If $i > 1$, then $x \in \blacktriangleright(\Gamma)(i) = \Gamma(i-1)$, and thus $a_{i-1}(x) \in A(i-1, x) = \blacktriangleright_\Gamma(A)(i,x)$ is well-defined, and if instead $i = 1$ then $0 \in \{0\} = 1 = \blacktriangleright_\Gamma(A)(i,x)$, implying that $\blacktriangleright_{\Gamma,A}(a)_i$ is well-defined. It holds for all $\Gamma \in \mathbf{S}_0$, $A \in \mathrm{Ty}(\Gamma)$, $a \in \mathrm{Tm}(\Gamma, A)$, $j \leq i$, and $x \in \blacktriangleright(\Gamma)(i)$ if $j > 1$ that

$$\begin{aligned} (\blacktriangleright_\Gamma(A)(j \leq i, x) \circ \blacktriangleright_{\Gamma,A}(a)_i)(x) &= (A(j-1 \leq i-1, x) \circ a_{i-1})(x)) \\ &= (a_{j-1} \circ \Gamma(j-1 \leq i-1))(x) \\ &= (\blacktriangleright_{\Gamma,A}(a)_j \circ \blacktriangleright(\Gamma)(j \leq i))(x), \end{aligned}$$

and if $j = 1$ that

$$\begin{aligned} (\blacktriangleright_\Gamma(A)(j \leq i, x) \circ \blacktriangleright_{\Gamma,A}(a)_i)(x) &= !_{\blacktriangleright_\Gamma(A)(i,x)}(\blacktriangleright_{\Gamma,A}(a)_i)(x) \\ &= 0 \\ &= (\blacktriangleright_{\Gamma,A}(a)_j \circ \blacktriangleright(\Gamma)(j \leq i))(x), \end{aligned}$$

and thus $\blacktriangleright_{\Gamma,A}(a) \in \mathrm{Tm}(\blacktriangleright(\Gamma), \blacktriangleright_\Gamma(A))$. It holds for all $\Gamma, \Delta \in \mathbf{S}_0$, $A \in \mathrm{Ty}(\Gamma)$, $\gamma \in \mathrm{Hom}_{\mathbf{S}}(\Delta, \Gamma)$, $a \in \mathrm{Tm}(\Gamma, A)$, $i \in \omega_0$, and $x \in \blacktriangleright(\Delta)(i)$ if $i > 1$ that

$$\begin{aligned} (\blacktriangleright_{\Delta,A[\gamma]} \circ \mathrm{Tm}(\gamma, A))(a)_i(x) &= \mathrm{Tm}(\gamma, A)(a)_{i-1}(x) \\ &= (a_{i-1} \circ \gamma_{i-1})(x) \\ &= (\blacktriangleright_{\Gamma,A}(a)_i \circ \blacktriangleright(\gamma)_i)(x) \\ &= (\mathrm{Tm}(\blacktriangleright(\gamma), \blacktriangleright_\Gamma(A)) \circ \blacktriangleright_{\Gamma,A})(a)_i(x) \\ &= \left(\left(\mathrm{Tm} \circ \int_{\blacktriangleright} (\blacktriangleright_-)^{\mathrm{op}}\right)(\gamma, A) \circ \blacktriangleright_{\Gamma,A}\right)(a)_i(x), \end{aligned}$$

and if $i = 1$ that

$$\begin{aligned} (\blacktriangleright_{\Delta,A[\gamma]} \circ \mathrm{Tm}(\gamma, A))(a)_i(x) &= 0 \\ &= (\blacktriangleright_{\Gamma,A}(a)_i \circ \blacktriangleright(\gamma)_i)(x) \\ &= (\mathrm{Tm}(\blacktriangleright(\gamma), \blacktriangleright_\Gamma(A)) \circ \blacktriangleright_{\Gamma,A})(a)_i(x) \\ &= \left(\left(\mathrm{Tm} \circ \int_{\blacktriangleright} (\blacktriangleright_-)^{\mathrm{op}}\right)(\gamma, A) \circ \blacktriangleright_{\blacktriangleright,A}\right)(a)_i(x), \end{aligned}$$

and thus $\blacktriangleright_{-,-} \colon \mathrm{Tm} \to \mathrm{Tm} \circ \int_{\blacktriangleright}(\blacktriangleright_-)^{\mathrm{op}}$ is a natural transformation.

Let $(\Gamma, A) \in \int_{\mathbf{S}}(\mathrm{Ty})_0$. We remind ourselves that

$$\begin{aligned} \Gamma \cdot A \colon \omega^{\mathrm{op}} &\to \mathbf{Set} \\ i &\mapsto \sum_{x \in \Gamma(i)} A(i, x) \\ j \leq i &\mapsto ((x, r) \mapsto (\Gamma(j \leq i)(x), A(j \leq i, x)(r))), \end{aligned}$$

and that for $\Delta \in \mathbf{S}_0$, $\gamma \in \mathrm{Hom}_{\mathbf{S}}(\Delta, \Gamma)$, $a \in \mathrm{Tm}(\Delta, A[\gamma])$, and $i \in \omega_0$

$$\begin{aligned} (\Phi^{-1}_{\Gamma,A})_\Delta(\gamma, a)_i \colon \Delta(i) &\to (\Gamma \cdot A)(i) \\ x &\mapsto (\gamma_i(x), a_i(x)). \end{aligned}$$

By the proof of Lemma 3.27, it holds for $i \in \omega_0$ and $(x,r) \in (\Gamma \cdot A)(i)$ that $(p_{\Gamma,A})_i(x,r)$ and $(q_{\Gamma,A})_i(x,r)$ are respectively the first and second coordinate of $(\mathrm{Id}_{\Gamma \cdot A})_i(x,r)$, i.e.

$$\begin{aligned} (p_{\Gamma,A})_i \colon (\Gamma \cdot A)(i) &\to \Gamma(i) \\ (x,r) &\mapsto x, \end{aligned} \qquad \begin{aligned} (q_{\Gamma,A})_i \colon (\Gamma \cdot A)(i) &\to A[p_{\Gamma,A}](i,-) \\ (x,r) &\mapsto r, \end{aligned}$$

where we note that $A[p_{\Gamma,A}](i,(x,r)) = A(i, p_{\Gamma,A}(x,r)) = A(i,x)$. This implies for $i \in \omega_0$ that

$$\begin{aligned} \blacktriangleright(p_{\Gamma,A})_i \colon \blacktriangleright(\Gamma \cdot A)(i) &\to \blacktriangleright(\Gamma)(i) \\ (x,r) &\mapsto x \\ 0 &\mapsto 0, \end{aligned} \quad \begin{aligned} \blacktriangleright_{\Gamma\cdot A, A[p_{\Gamma,A}]}(q_{\Gamma,A})_i \colon \blacktriangleright(\Gamma \cdot A)(i) &\to \blacktriangleright_{\Gamma\cdot A}(A[p_{\Gamma,A}])(i,-) \\ (x,r) &\mapsto r \\ 0 &\mapsto 0, \end{aligned}$$

where we note that by Lemma 3.14(i) that

$$\blacktriangleright_{\Gamma\cdot A}(A[p_{\Gamma,A}]) = \blacktriangleright_{\Gamma}(A)[\blacktriangleright(p_{\Gamma,A})],$$

and thus it holds for $i \in \omega_0$ that

$$\begin{aligned} \langle \blacktriangleright(p_{\Gamma,A}), \blacktriangleright_{\Gamma\cdot A, A[p_{\Gamma,A}]}(q_{\Gamma,A})\rangle_i \colon \blacktriangleright(\Gamma \cdot A)(i) &\to (\blacktriangleright(\Gamma) \cdot \blacktriangleright_{\Gamma}(A))(i) \\ (x,r) &\mapsto (\blacktriangleright(p_{\Gamma,A})(x,r), \blacktriangleright_{\Gamma\cdot A, A[p_{\Gamma,A}]}(x,r)) = (x,r) \\ 0 &\mapsto (\blacktriangleright(p_{\Gamma,A})(0), \blacktriangleright_{\Gamma\cdot A, A[p_{\Gamma,A}]}(0)) = (0,0), \end{aligned}$$

which is an isomorphism, and thus $\blacktriangleright$ extends to a weak CwF-morphism. ☐

Proposition 4.6. *Define the functor* $\blacktriangleleft \colon \mathbf{S} \to \mathbf{S}$ *by* $\blacktriangleleft(\Gamma)(i) = \Gamma(i+1)$, $\blacktriangleleft(\Gamma)(j \leq i) = \Gamma(j+1 \leq i+1)$, *and* $\blacktriangleleft(\gamma)_i = \gamma_{i+1}$ *for* $\Gamma, \Delta \in \mathbf{S}_0$, $\gamma \in \mathrm{Hom}_{\mathbf{S}}(\Delta, \Gamma)$, *and* $j \leq i \in \omega_1$. *Then* $\blacktriangleleft \dashv \blacktriangleright$.

Proof. Define for $\Gamma, \Delta \in \mathbf{S}_0$ $\varphi_{\Delta,\Gamma} \colon \mathrm{Hom}_{\mathbf{S}}(\blacktriangleleft(\Delta), \Gamma) \to \mathrm{Hom}_{\mathbf{S}}(\Delta, \blacktriangleright(\Gamma))$ by taking

$$\begin{array}{ccccccc} \Delta(2) & \xleftarrow{\Delta(2\leq 3)} & \Delta(3) & \xleftarrow{\Delta(3\leq 4)} & \Delta(4) & \xleftarrow{\Delta(4\leq 5)} & \cdots \\ \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow{\scriptstyle \gamma_3} & & \\ \Gamma(1) & \xleftarrow{\Gamma(1\leq 2)} & \Gamma(2) & \xleftarrow{\Gamma(2\leq 3)} & \Gamma(3) & \xleftarrow{\Gamma(3\leq 4)} & \cdots \end{array}.$$

to

$$\begin{array}{ccccccc} \Delta(1) & \xleftarrow{\Delta(1\leq 2)} & \Delta(2) & \xleftarrow{\Delta(2\leq 3)} & \Delta(3) & \xleftarrow{\Delta(3\leq 4)} & \cdots \\ \downarrow{\scriptstyle !_1} & & \downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \gamma_2} & & \\ 1 & \xleftarrow{!_{\Gamma(1)}} & \Gamma(1) & \xleftarrow{\Gamma(1\leq 2)} & \Gamma(2) & \xleftarrow{\Gamma(2\leq 3)} & \cdots \end{array}.$$

Since this is clearly both an isomorphism and natural in Δ and Γ, we have a natural isomorphism, implying the adjunction. ☐

Corollary 4.7. *Let $\mathcal{U}$ be a non-empty Grothendieck universe, and consider* $\mathbf{S}$ *as a category with families via Theorem 3.28. Then the CwF-structure on* $\mathbf{S}$ *extends to a CwDRA-structure with* $\blacktriangleleft$ *as the endofunctor.*

Proof. This follows from Proposition 4.5, Proposition 4.6, and Theorem 3.33. □

Let us explore how this CwDRA looks by following the definitions in Theorem 3.33. If we let $\varphi\colon \operatorname{Hom}_{\mathbf{S}}(\blacktriangleleft(-), -) \to \operatorname{Hom}_{\mathbf{S}}(-, \blacktriangleright(-))$ as in Proposition 4.6, then the component at $\Gamma \in \mathbf{S}_0$ of the unit of the adjunction is given by

$$\begin{aligned} (\eta_\Gamma)_i = \varphi_{\Gamma, \blacktriangleleft(\Gamma)}(\mathrm{Id}_{\blacktriangleleft(\Gamma)})_i \colon \Gamma(i) &\to \blacktriangleright(\blacktriangleleft(\Gamma))(i) \\ x &\mapsto \begin{cases} x & i > 1 \\ 0 & i = 1 \end{cases}. \end{aligned}$$

We then get for $\Gamma \in \mathbf{S}_0$ that

$$\begin{aligned} \blacktriangleright^*_\Gamma\colon \operatorname{Ty}(\blacktriangleleft(\Gamma)) &\to \operatorname{Ty}(\Gamma) \\ A &\mapsto \blacktriangleright_{\blacktriangleleft(\Gamma)}(A)[\eta_\Gamma], \end{aligned}$$

and thus it holds for $A \in \operatorname{Ty}(\blacktriangleleft(\Gamma))$, $j \le i \in \omega_1$, and $x \in \Gamma(i)$ that

$$\begin{aligned} \blacktriangleright^*_\Gamma(A)(i, x) &= \blacktriangleright_{\blacktriangleleft(\Gamma)}(A)[\eta_\Gamma](i, x) \\ &= \blacktriangleright_{\blacktriangleleft(\Gamma)}(A)(i, (\eta_\Gamma)_i(x)) \\ &= \begin{cases} A(i-1, x) & i > 1 \\ 1 & i = 1 \end{cases} \end{aligned}$$

and

$$\begin{aligned} \blacktriangleright^*_\Gamma(A)(j \le i, x) &= \blacktriangleright_{\blacktriangleleft(\Gamma)}(A)[\eta_\Gamma](j \le i, x) \\ &= \blacktriangleright_{\blacktriangleleft(\Gamma)}(A)(j \le i, (\eta_\Gamma)_i(x)) \\ &= \begin{cases} A(j-1 \le i-1, x) & j > 1 \\ !_{A(i-1,x)} & i > j = 1 \\ !_1 & i = 1 \end{cases}. \end{aligned}$$

Of relation to $\blacktriangleright$ we also define for each $\Gamma \in \mathbf{S}_0$ the morphism $\mathrm{next}_\Gamma \in \operatorname{Hom}_{\mathbf{S}}(\Gamma, \blacktriangleright(\Gamma))$ by

$$\begin{array}{ccccccc} \Gamma(1) & \xleftarrow{\Gamma(1\le 2)} & \Gamma(2) & \xleftarrow{\Gamma(2\le 3)} & \Gamma(3) & \xleftarrow{\Gamma(3\le 4)} & \cdots \\ {\scriptstyle !_{\Gamma(1)}}\big\downarrow & & {\scriptstyle \Gamma(1\le 2)}\big\downarrow & & {\scriptstyle \Gamma(2\le 3)}\big\downarrow & & \\ 1 & \xleftarrow{!_{\Gamma(1)}} & \Gamma(1) & \xleftarrow{\Gamma(1\le 2)} & \Gamma(2) & \xleftarrow{\Gamma(2\le 3)} & \cdots \end{array},$$

which is trivially commutative.

Lemma 4.8. $\mathrm{next}\colon \mathrm{Id}_{\mathbf{S}} \to \blacktriangleright$ *is a natural transformation.*

Proof. It holds for all $\Gamma, \Delta \in \mathbf{S}_0$, $\gamma \in \mathrm{Hom}_{\mathbf{S}}(\Delta, \Gamma)$, $i \in \omega_0$, and $x \in \Gamma(i)$ if $i > 1$ that

$$\begin{aligned}(\mathrm{next}_\Gamma \circ \mathrm{Id}_{\mathbf{S}}(\gamma))_i(\Gamma(i-1 \leq i) \circ \gamma_i \\ &= \gamma_{i-1} \circ \Delta(i-1 \leq i) \\ &= (\blacktriangleright(\gamma) \circ \mathrm{next}_\Delta)_i,\end{aligned}$$

and if $i = 1$ that

$$\begin{aligned}(\mathrm{next}_\Gamma \circ \mathrm{Id}_{\mathbf{S}}(\gamma))_i &= !_{\Gamma(1)} \circ \gamma_1 \\ &= !_{\Delta(1)} \\ &= !_1 \circ !_{\Delta(1)} \\ &= (\blacktriangleright(\gamma) \circ \mathrm{next}_\Delta)_i,\end{aligned}$$

and thus $\mathrm{next} \colon \mathrm{Id}_{\mathbf{S}} \to \blacktriangleright$ is a natural transformation. ☐

4.3 Contractive Morphisms and Fixpoints

Banach's fixed-pint theorem is an important theorem for metric spaces stating that any contractive function on a complete metric space has a unique fixpoint. We will here see a similar result in $\mathbf{S}$, stating that any contractive morphism has a uniquely determined fixpoint.

Definition 4.9. Let $\Gamma, \Delta \in \mathbf{S}_0$, and let $\gamma \in \mathrm{Hom}_{\mathbf{S}}(\Delta, \Gamma)$. We will then say that γ is contractive if there exists a $\delta \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Delta), \Gamma)$ such that $\gamma = \delta \circ \mathrm{next}_\Delta$, and we will call δ a witness of (the contractivity) of γ.

Definition 4.10. Let $\Gamma, \Delta, \Lambda \in \mathbf{S}_0$, and let $\gamma \in \mathrm{Hom}_{\mathbf{S}}(\Delta \times \Lambda, \Gamma)$. We will then say that γ is contractive in the first variable if there exists a $\delta \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Delta) \times \Lambda, \Gamma)$ such that $\gamma = \delta \circ (\mathrm{next}_\Delta \times \mathrm{Id}_\Lambda)$, and we will cal δ a witness of(the contractivity) of γ.

There is in general no guarantee that witnesses be unique as next_Γ is not generally epic. However, if $\Gamma(i \leq i+1)$ is surjective for all $i \in \omega_0$ (Γ is then said to be total), next_Γ is an epimorphism, and thus witnesses do become unique[1].

Let us briefly remind ourselves about exponentials in $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ for a small category $\mathbf{C}$. It holds by the Yoneda Lemma for any $X, Y \in \mathbf{Set}_0^{\mathbf{C}^{\mathrm{op}}}$ and $C \in \mathbf{C}_0$ that

$$Y^X(C) \cong \mathrm{Hom}_{\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}\left(\mathrm{Hom}_{\mathbf{C}}(-, C), Y^X\right) \cong \mathrm{Hom}_{\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}(\mathrm{Hom}_{\mathbf{C}}(-, C) \times X, Y),$$

and one therefore generally defines

$$\begin{aligned}Y^X \colon \mathbf{C}^{\mathrm{op}} &\to \mathbf{Set} \\ C &\mapsto \mathrm{Hom}_{\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}(\mathrm{Hom}_{\mathbf{C}}(-, C) \times X, Y) \\ f &\mapsto \mathrm{Hom}_{\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}(\mathrm{Hom}_{\mathbf{C}}(-, f) \times \mathrm{Id}_X, Y),\end{aligned}$$

[1]Interestingly, it can also be proven that each $\Gamma(i \leq i+1)$ being injective also guarantess uniqueness of witnesses. Given that the first i components of the witness are defined, can you determine the exact condition for component $i+1$ being uniquely defined?

which is indeed an exponential. For $X, Y, A \in \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ the evaluation morphism $\epsilon \in \mathrm{Hom}_{\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}(Y^X \times X, Y)$ has component at $C \in \mathbf{C}_0$

$$\begin{aligned} \epsilon_C \colon \mathrm{Hom}_{\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}(\mathrm{Hom}_{\mathbf{C}}(-, C) \times X, Y) \times X(C) &\to Y(C) \\ (\delta, x) &\mapsto \delta_C(\mathrm{Id}_C, x), \end{aligned}$$

and the exponential transpose of $\gamma \in \mathrm{Hom}_{\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}(A \times X, Y)$ has component at $C \in \mathbf{C}_0$

$$\widetilde{\gamma}_C \colon A(C) \to \mathrm{Hom}_{\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}}(\mathrm{Hom}_{\mathbf{C}}(-, C) \times X, Y)$$

given for $a \in A(C)$ and $D \in \mathbf{C}_0$ by

$$\begin{aligned} \widetilde{\gamma}_C(a)_D \colon \mathrm{Hom}_{\mathbf{C}}(D, C) \times X(D) &\to Y(D) \\ (f, x) &\mapsto \gamma_D(A(f)(a), x). \end{aligned}$$

Since $\mathbf{S} = \mathbf{Set}^{\omega^{\mathrm{op}}}$ this all holds in $\mathbf{S}$, though we will use the notation $\Delta \to \Gamma$ for $\Gamma, \Delta \in \mathbf{S}_0$ instead of Γ^{Δ}.

Lemma 4.11. *Let $\Gamma, \Gamma', \Delta, \Delta', \Lambda \in \mathbf{S}_0$. It then holds that*

(i) If $\gamma \in \mathrm{Hom}_{\mathbf{S}}(\Delta, \Gamma)$, $\delta \in \mathrm{Hom}_{\mathbf{S}}(\Lambda, \Delta)$, and either is contractive then $\gamma \circ \delta$ is contractive.

(ii) If $\gamma \in \mathrm{Hom}_{\mathbf{S}}(\Delta, \Gamma)$ and $\gamma' \in \mathrm{Hom}_{\mathbf{S}}(\Delta', \Gamma')$ are both contractive then $\gamma \times \gamma'$ is contractive.

(iii) $\gamma \in \mathrm{Hom}_{\mathbf{S}}(\Lambda \times \Delta, \Gamma)$ is contractive in the first variable if and only if the exponential transpose $\widetilde{\gamma} \in \mathrm{Hom}_{\mathbf{C}}(\Lambda, \Delta \to \Gamma)$ is contractive.

Proof. (i): Assume first that δ is contractive with witness $\lambda \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Lambda), \Delta)$. It then holds that $\gamma \circ \lambda \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Lambda), \Gamma)$ and

$$(\gamma \circ \lambda) \circ \mathrm{next}_{\Lambda} = \gamma \circ (\lambda \circ \mathrm{next}_{\Lambda}) = \gamma \circ \delta,$$

and thus $\gamma \circ \delta$ is contractive. Assume instead that γ is contractive with witness $\lambda \in \mathrm{Hom}_{\mathbf{S}} \blacktriangleright (\Delta), \Gamma)$. It then holds by Lemma 4.8 that $\lambda \circ \blacktriangleright(\delta) \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Lambda), \Gamma)$ and

$$(\lambda \circ \blacktriangleright(\delta)) \circ \mathrm{next}_{\Lambda} = \lambda \circ (\blacktriangleright(\delta) \circ \mathrm{next}_{\Lambda}) = \lambda \circ (\mathrm{next}_{\Delta} \circ \delta) = (\lambda \circ \mathrm{next}_{\Delta}) \circ \delta = \gamma \circ \delta,$$

and thus $\gamma \circ \delta$ is contractive.

(ii): Let $\lambda \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Delta), \Gamma)$ and $\lambda' \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Delta'), \Gamma')$ be witness of respectively γ and γ'. We have an isomorphism $\varphi \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Delta \times \Delta'), \blacktriangleright(\Delta) \times \blacktriangleright(\Delta'))$ given by $\varphi_i = \mathrm{Id}_{\Delta(i-1) \times \Delta'(i-1)}$ for $i \in \omega_0 \backslash \{0\}$ and $\varphi_1(0) = (0, 0)$. It holds for $i \in \omega_0$ and $(x, x') \in \Delta(i) \times \Delta'(i)$ if $i > 1$ that

$$(\varphi \circ \mathrm{next}_{\Delta \times \Delta'})_i(x, x') = (\Delta(i-1 \leq i)(x) \times \Delta'(i-1 \leq i)(x')) = (\mathrm{next}_{\Delta} \times \mathrm{next}_{\Delta'})_i(x, x')$$

and if $i = 1$ that

$$(\varphi \circ \mathrm{next}_{\Delta \times \Delta'})_i(x, x') = \varphi_i(0) = (0, 0) = (\mathrm{next}_{\Delta} \times \mathrm{next}_{\Delta'})_i(x, x'),$$

and thus $\varphi \circ \mathrm{next}_{\Delta \times \Delta'} = \mathrm{next}_{\Delta} \times \mathrm{next}_{\Delta'}$, implying that

$$\begin{aligned} ((\lambda \times \lambda') \circ \varphi) \circ \mathrm{next}_{\Delta \times \Delta'} &= (\lambda \times \lambda') \circ (\varphi \circ \mathrm{next}_{\Delta \times \Delta'}) \\ &= (\lambda \times \lambda') \circ (\mathrm{next}_{\Delta} \times \mathrm{next}_{\Delta'}) \\ &= (\lambda \circ \mathrm{next}_{\Delta}) \times (\lambda' \circ \mathrm{next}_{\Delta'}) \\ &= \gamma \times \gamma', \end{aligned}$$

and thus $\gamma \times \gamma'$ is contractive.

(iii): Assume first that γ is contractive in the first variable, with witness $\lambda \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Lambda) \times \Delta, \Gamma)$. We then have the exponential transpose $\widetilde{\lambda} \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Lambda), \Delta \to \Gamma)$, which we wish to prove satisfies $\widetilde{\lambda} \circ \mathrm{next}_\Lambda = \widetilde{\gamma}$. If $\epsilon \in \mathrm{Hom}_{\mathbf{S}}((\Delta \to \Gamma) \times \Delta, \Gamma)$ is the evaluation morphism of Γ^Δ, we get that

$$\begin{aligned}
\epsilon \circ \left(\left(\widetilde{\lambda} \circ \mathrm{next}_\Lambda\right) \times \mathrm{Id}_\Delta\right) &= \epsilon \circ \left(\left(\widetilde{\lambda} \circ \mathrm{next}_\Lambda\right) \times (\mathrm{Id}_\Delta \circ \mathrm{Id}_\Delta)\right) \\
&= \epsilon \circ (\widetilde{\lambda} \times \mathrm{Id}_\Delta) \circ (\mathrm{next}_\Lambda \times \mathrm{Id}_\Delta) \\
&= \lambda \circ (\mathrm{next}_\Lambda \times \mathrm{Id}_\Delta) \\
&= \gamma,
\end{aligned}$$

and thus by the uniqueness of the exponential transpose, we have that $\widetilde{\lambda} \circ \mathrm{next}_\Lambda = \widetilde{\gamma}$, implying that $\widetilde{\gamma}$ is contractive. Assume instead that $\widetilde{\gamma}$ is contractive, and choose $\lambda \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Lambda), \Delta \to \Gamma)$ such that $\widetilde{\gamma} = \lambda \circ \mathrm{next}_\Lambda$. Let $\overline{\lambda} = \epsilon \circ (\lambda \times \mathrm{Id}_\Delta) \in \mathrm{Hom}_{\mathbf{Set}}(\blacktriangleright(\Lambda) \times \Delta, \Gamma)$. It then holds that

$$\begin{aligned}
\overline{\lambda} \circ (\mathrm{next}_\Lambda \times \mathrm{Id}_\Delta) &= \varepsilon \circ (\lambda \times \mathrm{Id}_\Delta) \circ (\mathrm{next}_\Lambda \times \mathrm{Id}_\Delta) \\
&= \epsilon \circ ((\lambda \circ \mathrm{next}_\Lambda) \times (\mathrm{Id}_\Delta \circ \mathrm{Id}_\Delta)) \\
&= \epsilon \circ (\widetilde{\gamma} \times \mathrm{Id}_\Delta) \\
&= \gamma,
\end{aligned}$$

and thus γ is contractive in the first variable. □

We will now show how contractive morphisms give rise to fixpoints. There are several equivalent ways to view fixpoints; we will start giving the intuition behind fixed global points, and we will afterwards show how to use this to construct a fixpoint morphism.

Let $\Gamma \in \mathbf{S}_0$, let $\gamma \in \mathrm{Hom}_{\mathbf{S}}(\Gamma, \Gamma)$ be contractive, and let $\delta \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Gamma), \Gamma)$ such that $\gamma = \delta \circ \mathrm{next}_\Gamma$. We will find a $\varpi \in \mathrm{Hom}_{\mathbf{S}}(1, \Gamma)$ such that $\gamma \circ \varpi = \varpi$. This gives rise to the diagram

$$\begin{array}{ccccccc}
1 & \xleftarrow{\mathrm{Id}_1} & 1 & \xleftarrow{\mathrm{Id}_1} & 1 & \xleftarrow{\mathrm{Id}_1} & \cdots \\
\downarrow{\scriptstyle \varpi_1} & & \downarrow{\scriptstyle \varpi_2} & & \downarrow{\scriptstyle \varpi_3} & & \\
\Gamma(1) & \xleftarrow{\Gamma(1\le 2)} & \Gamma(2) & \xleftarrow{\Gamma(2\le 3)} & \Gamma(3) & \xleftarrow{\Gamma(3\le 4)} & \cdots \\
\downarrow{\scriptstyle !_{\Gamma(1)}} & & \downarrow{\scriptstyle \Gamma(1\le 2)} & & \downarrow{\scriptstyle \Gamma(2\le 3)} & & \\
1 & \xleftarrow{!_{\Gamma(1)}} & \Gamma(1) & \xleftarrow{\Gamma(1\le 2)} & \Gamma(2) & \xleftarrow{\Gamma(2\le 3)} & \cdots \\
\downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \delta_3} & & \\
\Gamma(1) & \xleftarrow{\Gamma(1\le 2)} & \Gamma(2) & \xleftarrow{\Gamma(3\le 3)} & \Gamma(3) & \xleftarrow{\Gamma(3\le 4)} & \cdots
\end{array} .$$

The key to constructing a fixpoint is to note that each set in the third row has a distinguished point, which we can use to construct the fixpoint in the column by application of δ. In the first column, $\blacktriangleright(\Gamma)(1) = 1$ contains only the point 0, and we therefore define $\varpi_1(0) = \delta_1(0)$. If we have defined $\varpi_i(0)$, this becomes a distinguished point in $\Gamma(i) = \blacktriangleright(\Gamma)(i+1)$, and thus we may define $\varpi_{i+1}(0) = \delta_{i+1}(\varpi_i(0))$. A simple calculation now shows that these points are indeed fixpoints, that ϖ is natural, and that ϖ is the unique fixed global point of f.

Since the point is unique, we can define this a function from the contractive endomorphisms on Γ to the global points of Γ, but since a contractive morphism $\gamma \in \mathrm{Hom}_{\mathbf{S}}(\Gamma, \Gamma)$ is uniquely determined by the corresponding $\delta \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Gamma), \Gamma)$, we can also consider this fixpoint function as going from $\mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Gamma), \Gamma)$ to $\mathrm{Hom}_{\mathbf{S}}(1, \Gamma)$. If we let $\pi_X \in \mathrm{Hom}_{\mathbf{S}}(X \times 1, X)$ be the canonical projection, the function taking $g \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Gamma), \Gamma)$ to $\widetilde{\delta \circ \pi_{\blacktriangleright(X)}} \in \mathrm{Hom}_{\mathbf{S}}(1, \blacktriangleright$

$(\Gamma) \to \Gamma)$ is an isomorphism, and we can thus define a function taking a global point of $\blacktriangleright(\Gamma) \to \Gamma)$ to a global point of Γ. One potential way to define such a function, is to give a morphism $\mathrm{fix}_\Gamma \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Gamma) \to \Gamma, \Gamma)$ such that the function is $\mathrm{Hom}_{\mathbf{S}}(1, \mathrm{fix}_\Gamma)$. This is the approach we will take in the following theorem, which also generalizes the above discussion to morphisms, which are contractive in the first variable.

Theorem 4.12. *For all $\Gamma \in \mathbf{S}_0$ there exists a morphism $\mathrm{fix}_\Gamma \colon (\blacktriangleright(\Gamma) \to \Gamma) \to \Gamma$ such that for all $\Delta \in \mathbf{S}_0$, $\gamma \in \mathrm{Hom}_{\mathbf{S}}(\Gamma \times \Delta, \Gamma)$ contractive in the first variable with witness $\delta \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Gamma) \times \Delta, \Gamma)$ it holds that $\mathrm{fix}_\Gamma \circ \widetilde{\delta}$, where $\widetilde{\delta} \in \mathrm{Hom}_{\mathbf{S}}(\Delta, \blacktriangleright(\Gamma) \to \Gamma)$ is the exponential transpose, is the unique $\lambda \in \mathrm{Hom}_{\mathbf{S}}(\Delta, \Gamma)$ such that $\gamma \circ \langle \lambda, \mathrm{Id}_\Delta \rangle = \lambda$, where $\langle \lambda, \mathrm{Id}_\Delta \rangle$ is the mediating morphism of the product $\Lambda \times \Delta$.*

Proof. Define for each $i \in \omega_0$

$$
\begin{aligned}
(\mathrm{fix}_\Gamma)_i \colon (\blacktriangleright(\Gamma) \to \Gamma)(i) &\to \Gamma(i) \\
\alpha &\mapsto \begin{cases} \alpha_i(i \leq i, 0) & i = 1 \\ \alpha_i(i \leq i, (\mathrm{fix}_\Gamma)_{i-1}(\alpha|_{i-1})) & i > 1 \end{cases}.
\end{aligned}
$$

It holds for all $j \leq i \in \omega_1$ and $\alpha \in (\blacktriangleright(\Gamma) \to \Gamma)(i)$ if $i = j$ that

$$
\begin{aligned}
(\mathrm{fix}_\Gamma)_j \circ (\blacktriangleright(\Gamma) \to \Gamma)(j \leq i) &= (\mathrm{fix}_\Gamma)_j \circ \mathrm{Id}_{(\blacktriangleright(\Gamma) \to \Gamma)(j)} \\
&= \mathrm{Id}_{\Gamma(i)} \circ (\mathrm{fix}_\Gamma)_i \\
&= \Gamma(j \leq i) \circ (\mathrm{fix}_\Gamma)_i,
\end{aligned}
$$

if $j = 1 < i$ that

$$
\begin{aligned}
((\mathrm{fix}_\Gamma)_j \circ (\blacktriangleright(\Gamma) \to \Gamma)(j \leq i))(\alpha) &= (\mathrm{fix}_\Gamma)_j(\alpha \circ (\mathrm{Hom}_\omega(-, j \leq i) \times \mathrm{Id}_{\blacktriangleright(\Gamma)})) \\
&= (\alpha \circ (\mathrm{Hom}_\omega(-, j \leq i) \times \mathrm{Id}_{\blacktriangleright(\Gamma)}))_j(j \leq j, 0) \\
&= (\alpha_j \circ (\mathrm{Hom}_\omega(j, j \leq i) \times \mathrm{Id}_{\blacktriangleright(\Gamma)(j)}))(j \leq j, 0) \\
&= \alpha_j(j \leq i, 0) \\
&= \alpha_j(j \leq i, (\blacktriangleright(\Gamma)(j \leq i) \circ (\mathrm{fix}_\Gamma)_{i-1})(\alpha|_{i-1})) \\
&= (\alpha_j \circ (\mathrm{Hom}_\omega(j \leq i, i) \times \blacktriangleright(\Gamma)(j \leq i)))(i \leq i, (\mathrm{fix}_\Gamma)_{i-1}(\alpha|_{i-1})) \\
&= (\alpha_j \circ (\mathrm{Hom}_\omega(-, i) \times \blacktriangleright(\Gamma))(j \leq i))(i \leq i, (\mathrm{fix}_\Gamma)_{i-1}(\alpha|_{i-1})) \\
&= (\Gamma(j \leq i) \circ \alpha_i)(i \leq i, (\mathrm{fix}_\Gamma)_{i-1}(\alpha|_{i-1})) \\
&= (\Gamma(j \leq i) \circ (\mathrm{fix}_\Gamma)_i)(\alpha),
\end{aligned}
$$

and if $1 < j < i$ and $(\mathrm{fix}_\Gamma)_{j-1} \circ (\blacktriangleright(\Gamma) \to \Gamma)(j-1 \leq i-1) = \Gamma(j-1 \leq i-1) \circ (\mathrm{fix}_\Gamma)_{i-1}$ that

$$\begin{aligned}
&((\mathrm{fix}_\Gamma)_j \circ (\blacktriangleright(\Gamma) \to \Gamma)(j \leq i))(\alpha) \\
&= ((\blacktriangleright(\Gamma) \to \Gamma)(j \leq i)(\alpha))_j(j \leq j, (\mathrm{fix}_\Gamma)_{j-1}((\blacktriangleright(\Gamma) \to \Gamma)(j \leq i)(\alpha)|_{j-1})) \\
&= (\alpha \circ (\mathrm{Hom}_\omega(-, j \leq i) \times \mathrm{Id}_{\blacktriangleright(\Gamma)}))_j(j \leq j, (\mathrm{fix}_\Gamma)_{j-1}(\alpha|_{j-1})) \\
&= (\alpha_j \circ (\mathrm{Hom}_\omega(j, j \leq i) \times \mathrm{Id}_{\blacktriangleright(\Gamma)(j)}))(j \leq j, (\mathrm{fix}_\Gamma)_{j-1}(\alpha|_{j-1})) \\
&= \alpha_j(j \leq i, (\mathrm{fix}_\Gamma)_{j-1}(\alpha|_{j-1})) \\
&= \alpha_j(j \leq i, ((\mathrm{fix}_\Gamma)_{j-1} \circ (\blacktriangleright(\Gamma) \to \Gamma)(j-1 \leq i-1))(\alpha|_{i-1})) \\
&= \alpha_j(j \leq i, (\Gamma(j-1 \leq i-1) \circ (\mathrm{fix}_\Gamma)_{i-1})(\alpha|_{i-1})) \\
&= \alpha_j(j \leq i, (\blacktriangleright(\Gamma)(j \leq i) \circ (\mathrm{fix}_\Gamma)_{i-1})(\alpha|_{i-1})) \\
&= (\alpha_j \circ (\mathrm{Hom}_\omega(j \leq i, i) \times \blacktriangleright(\Gamma)(j \leq i)))(i \leq i, (\mathrm{fix}_\Gamma)_{i-1}(\alpha|_{i-1})) \\
&= (\alpha_j \circ (\mathrm{Hom}_\omega(-, i) \times \blacktriangleright(\Gamma))(j \leq i))(i \leq i, (\mathrm{fix}_\Gamma)_{i-1}(\alpha|_{i-1})) \\
&= (\Gamma(j \leq i) \circ \alpha_i)(i \leq i, (\mathrm{fix}_\Gamma)_{i-1}(\alpha|_{i-1})) \\
&= (\Gamma(j \leq i) \circ (\mathrm{fix}_\Gamma)_i)(\alpha),
\end{aligned}$$

and thus by induction $\mathrm{fix}_\Gamma \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright(\Gamma) \to \Gamma, \Gamma)$ is a natural transformation.

It holds for all $i \in \omega_0$ and $x \in \Delta(i)$ if $i = 1$ that

$$\begin{aligned}
(\gamma \circ \langle \mathrm{fix}_\Gamma \circ \widetilde{\delta}, \mathrm{Id}_\Delta \rangle)_i(x) &= \gamma_i(((\mathrm{fix}_\Gamma)_i \circ \widetilde{\delta}_i)(x), x) \\
&= \gamma_i(\widetilde{\delta}_i(x)_i(i \leq i, 0), x) \\
&= (\delta \circ (\mathrm{next}_\Gamma \times \mathrm{Id}_\Delta))_i(\widetilde{\delta}_i(x)_i(i \leq i, 0), x) \\
&= \delta_i(0, x) \\
&= \delta_i(0, \Delta(i \leq i)(x)) \\
&= \widetilde{\delta}_i(x)_i(i \leq i, 0) \\
&= (\mathrm{fix}_\Gamma \circ \widetilde{\delta})_i(x),
\end{aligned}$$

and if $i > 1$ and $(\gamma \circ \langle \mathrm{fix}_\Gamma \circ \widetilde{\delta}, \mathrm{Id}_\Delta \rangle)_{i-1} = (\mathrm{fix}_\Gamma \circ \widetilde{\delta})_{i-1}$ that

$$\begin{aligned}
(\gamma \circ \langle \mathrm{fix}_\Gamma \circ \widetilde{\delta}, \mathrm{Id}_\Delta \rangle)_i(x) &= \gamma_i(((\mathrm{fix}_\Gamma)_i \circ \widetilde{\delta}_i)(x), x) \\
&= (\delta \circ (\mathrm{next}_\Gamma \times \mathrm{Id}_\Delta))_i(((\mathrm{fix}_\Gamma)_i \circ \widetilde{\delta}_i)(x), x) \\
&= \delta_i((\mathrm{next}_\Gamma \circ \mathrm{fix}_\Gamma \circ \widetilde{\delta})_i(x), x) \\
&= \delta_i((\mathrm{next}_\Gamma \circ \widetilde{\delta}_i(x))_i(i \leq i, (\mathrm{fix}_\Gamma)_{i-1}(\widetilde{\delta}_i(x)|_{i-1})), x) \\
&= \delta_i((\mathrm{next}_\Gamma \circ \delta)_i((\mathrm{fix}_\Gamma)_{i-1}(\widetilde{\delta}_i(x)|_{i-1}), \Delta(i \leq i)(x)), x) \\
&= \delta_i((\Gamma(i-1 \leq i) \circ \delta_i)((\mathrm{fix}_\Gamma)_{i-1}(\widetilde{\delta}_i(x)|_{i-1}), x), x) \\
&= \delta_i((\delta_{i-1} \circ (\blacktriangleright(\Gamma) \times \Delta)(i-1 \leq i))((\mathrm{fix}_\Gamma)_{i-1}(\widetilde{\delta}_i(x)|_{i-1}), x), x) \\
&= \delta_i(\delta_{i-1}((\blacktriangleright(\Gamma)(i-1 \leq i) \circ (\mathrm{fix}_\Gamma)_{i-1})(\widetilde{\delta}_i(x)|_{i-1}), \Delta(i-1 \leq i)(x)), x) \\
&= \delta_i(\delta_{i-1}((\mathrm{next}_\Gamma \circ \mathrm{fix}_\Gamma)_{i-1}(\widetilde{\delta}_i(x)|_{i-1}), x|_{i-1}), x) \\
&= \delta_i((\delta \circ (\mathrm{next}_\Gamma \times \mathrm{Id}_\Delta))_{i-1}((\mathrm{fix}_\Gamma)_{i-1}(\widetilde{\delta}_i(x)|_{i-1}), x|_{i-1}), x) \\
&= \delta_i(\gamma_{i-1}((\mathrm{fix}_\Gamma)_{i-1}(\widetilde{\delta}_i(x)|_{i-1}), x|_{i-1}), x) \\
&= \delta_i(\gamma_{i-1}((\mathrm{fix}_\Gamma)_{i-1}(((\blacktriangleright(\Gamma) \to \Gamma)(i-1 \leq i) \circ \widetilde{\delta}_i)(x)), x|_{i-1}), x) \\
&= \delta_i(\gamma_{i-1}((\mathrm{fix}_\Gamma)_{i-1}((\widetilde{\delta}_{i-1} \circ \Delta(i-1 \leq i))(x)), x|_{i-1}), x) \\
&= \delta_i(\gamma_{i-1}((\mathrm{fix}_\Gamma \circ \widetilde{\delta})_{i-1}(x|_{i-1}), x|_{i-1}), x) \\
&= \delta_i((\gamma \circ \langle \mathrm{fix}_\Gamma \circ \widetilde{\delta}, \mathrm{Id}_\Delta \rangle)_{i-1}(x|_{i-1}), x) \\
&= \delta_i((\mathrm{fix}_\Gamma \circ \widetilde{\delta})_{i-1}(x|_{i-1}), x) \\
&= \delta_i((\mathrm{fix}_\Gamma)_{i-1}(\widetilde{\delta}_i(x)|_{i-1}), \Delta(i \leq i)(x)) \\
&= \widetilde{\delta}_i(x)_i(i \leq i, (\mathrm{fix}_\Gamma)_{i-1}(\widetilde{\delta}_i(x)|_{i-1})) \\
&= (\mathrm{fix}_\Gamma \circ \widetilde{\delta})_i(x),
\end{aligned}$$

and thus by induction $\gamma \circ \langle \mathrm{fix}_\Gamma \circ \widetilde{\delta}, \mathrm{Id}_\Delta \rangle = \mathrm{fix}_\Gamma \circ \widetilde{\delta}$.

Let $\lambda \in \mathrm{Hom}_{\mathbf{S}}(\Delta, \Gamma)$ with $\gamma \circ \langle \lambda, \mathrm{Id}_\Delta \rangle = \lambda$. It then holds $i \in \omega_0$ and $x \in \Delta(i)$ if $i = 1$ that

$$\begin{aligned}
\lambda_i(x) &= (\gamma \circ \langle \lambda, \mathrm{Id}_\Delta \rangle)_i(x) \\
&= (\delta \circ (\mathrm{next}_\Gamma \times \mathrm{Id}_\Delta) \circ \langle \lambda, \mathrm{Id}_\Delta \rangle)_i(x) \\
&= (\delta \circ \langle \mathrm{next}_\Gamma \circ \lambda, \mathrm{Id}_\Delta \rangle)_i(x) \\
&= \delta_i((\mathrm{next}_\Gamma \circ \lambda)_i(x), x) \\
&= \delta_i(0, x),
\end{aligned}$$

and if $i > 1$ that

$$\begin{aligned}
\lambda_i(x) &= (\gamma \circ \langle \lambda, \mathrm{Id}_\Delta \rangle)_i(x) \\
&= (\delta \circ (\mathrm{next}_\Gamma \times \mathrm{Id}_\Delta) \circ \langle \lambda, \mathrm{Id}_\Delta \rangle)_i(x) \\
&= (\delta \circ \langle \mathrm{next}_\Gamma \circ \lambda, \mathrm{Id}_\Delta \rangle)_i(x) \\
&= \delta_i((\mathrm{next}_\Gamma \circ \lambda)_i(x), x) \\
&= \delta_i((\Gamma(i-1 \leq i) \circ \lambda_i)(x), x) \\
&= \delta_i((\lambda_{i-1} \circ \Delta(i-1 \leq i))(x), x),
\end{aligned}$$

and thus by induction, λ is uniquely determined.

Finally, if we choose $\Delta = \blacktriangleright(\Gamma) \to \Gamma$ and $\widetilde{\delta} = \mathrm{Id}_{\blacktriangleright(\Gamma) \to \Gamma}$, $\widetilde{\delta}$ is epic, and thus since $\mathrm{fix}_\Gamma \circ \widetilde{\delta}$ is uniquely determined, fix_Γ is also unique. □

4.4 Example: Streams

We will now use streams as an example to see how to solve equations of types. A stream of natural numbers is an infinite sequence of natural numbers with two operations; a head operation, which returns the first element of the stream, and a tail operation, which is the rest of the stream, i.e. what the stream does later. We should therefore have some sort of equation $\texttt{Strm} \cong \texttt{N} \times \blacktriangleright \texttt{Strm}$, stating that a stream is pretty much what is happening now and what is happening later.

In order to implement this idea into $\mathbf{S}$, we will let $\mathcal{U}$ be a Grothendieck universe containing the set of natural numbers N[2], and consider $\mathbf{S}$ as a CwDRA via Corollary 4.7. We then define $\texttt{N} \in \mathrm{Ty}(1)$ as the presheaf on $\int_\omega(1)$ constantly equal to N and wish to find $\texttt{Strm} \in \mathrm{Ty}(1)$ such that $1 \cdot \texttt{Strm} \cong 1 \cdot \texttt{N} \times \blacktriangleright (1 \cdot \texttt{Strm})$[3]. Assuming such a type exists, let $\delta \in \mathrm{Hom}_{\mathbf{S}}(\blacktriangleright (1 \cdot \texttt{Strm}) \times 1 \cdot \texttt{N}, 1 \cdot \texttt{Strm})$ be an isomorphism. The first component of δ gives a bijection between $\blacktriangleright (1 \cdot \texttt{Strm})(1) \times (1 \cdot \texttt{N})(1) = 1 \times 1 \times N$ and $(1 \cdot \texttt{Strm})(1) = 1 \times \texttt{Strm}(1, 0)$, and thus we may choose $\texttt{Strm}(1, 0) = N$ and

$$\begin{aligned} \delta_1 \colon 1 \times 1 \times N &\to 1 \times N \\ (0, 0, n) &\mapsto (0, n), \end{aligned}$$

which is the unique choice up to isomorphism. Assuming we know $\texttt{Strm}(i)$ and δ_i for some $i \in \omega_0$, δ_{i+1} is a bijection between $\blacktriangleright (1 \cdot \texttt{Strm})(i+1) \times (1 \cdot \texttt{N})(i+1) = 1 \times \texttt{Strm}(i, 0) \times 1 \times N$ and $(1 \cdot \texttt{Strm})(i+1) = 1 \times \texttt{Strm}(i+1, 0)$, and we may thus choose $\texttt{Strm}(i+1, 0) = \texttt{Strm}(i, 0) \times N$ and

$$\begin{aligned} \delta_{i+1} \colon 1 \times \texttt{Strm}(i, 0) \times 1 \times N &\to 1 \times \texttt{Strm}(i, 0) \times N \\ (0, t, 0, n) &\mapsto (0, t, n), \end{aligned}$$

which again is the unique choice up to isomorphism. By induction, this shows that $\texttt{Strm}(i, 0) = N^i$ for $i \in \omega_0$, and naturality of δ implies that for $i \in \omega_0$ and $(n_1, \ldots, n_{i+1}) \in \texttt{Strm}(i+1, 0)$ that

$$\begin{aligned} &(1 \cdot \texttt{Strm})(i \le i+1)(0, n_1, \ldots, n_{i+1}) \\ &= (\delta_i \circ (\blacktriangleright (1 \cdot \texttt{Strm}) \times 1 \cdot \texttt{N})(i \le i+1) \circ \delta_{i+1}^{-1})(0, n_1, \ldots, n_{i+1}) \\ &= (\delta_i \circ (\blacktriangleright (1 \cdot \texttt{Strm})(i \le i+1) \times (1 \cdot \texttt{N})(i \le i+1))(0, n_1, \ldots, n_i, 0, n_{i+1}) \\ &= \delta_i(\blacktriangleright (1 \cdot \texttt{Strm})(i \le i+1)(0, n_1, \ldots, n_i), (1 \cdot \texttt{N})(i \le i+1)(0, n_{i+1})) \\ &= \delta_i(\blacktriangleright (1 \cdot \texttt{Strm})(i \le i+1)(0, n_1, \ldots, n_i), 0, n_{i+1}) \\ &= (\blacktriangleright (1 \cdot \texttt{Strm})(i \le i+1)(0, n_1, \ldots, n_i), n_{i+1}), \end{aligned}$$

and thus in particular

$$\begin{aligned} (1 \cdot \texttt{Strm})(1 \le 2)(0, n_1, n_2) &= (\blacktriangleright (1 \cdot \texttt{Strm})(1 \le 2)(0, n_1), n_2) \\ &= (0, n_2), \end{aligned}$$

which by induction implies that

$$(1 \cdot \texttt{Strm})(i \le i+1)(0, n_1, \ldots, n_{i+1}) = (0, n_2, \ldots, n_{i+1}).$$

There are more general methods for solving these kinds of equations, in particular in [2, Section 2.6] fixpoints of locally contractive functors are used, similarly to the fixpoints of contractive morphisms, and in [1] fixpoints on universes are used to the same effect.

[2] As discussed earlier, no such universe is guaranteed to exists by ZFC, but we will take as an additional axiom that such a Grothendieck universe does exists.

[3] We write the equation in terms of contexts rather than types, as we have not yet concerned ourselves with isomorphisms of types.

Bibliography

[1] Lars Birkedal and Rasmus Ejlers Møgelberg. "Intensional type theory with guarded recursive types qua fixed points on universes". English. In: *Annual Symposium on Logic in Computer Science* (2013), pp. 213–222.

[2] Lars Birkedal et al. "First Steps in Synthetic Guarded Domain Theory: Step-Indexing in the Topos of Trees". In: *Logical Methods in Computer Science* 8.4 (2012), pp. 1–45.

[3] Lars Birkedal et al. "Modal Dependent Type Theory and Dependent Right Adjoint". In: *CoRR* (2019). URL: `http://arxiv.org/abs/1804.05236`.

[4] V.A.J. Borghuis. "Coming to terms with modal logic : on the interpretation of modalities in typed lambda-calculus". PhD thesis. Department of Mathematics and Computer Science, 1994.

[5] Simon Castellan, Pierre Clairambault and Peter Dybjer. "Categories with Families. Unityped, Simply Typed, and Dependently Typed". In: *CORR* (2019). URL: `http://arxiv.org/abs/1904.00827`.

[6] Martin Hofmann. "Extensional Construct in Intensional Type Theory". In: Springer, 1997. Chap. Syntax and semantics of dependent types, pp. 13–54.

[7] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in Geometry and Logic. A First Introduction to Topos Theory.* Springer Verlag, 1994. ISBN: 978-0-387-97710-2.

[8] The Univalent Foundations Program. *Homotopy Type Theory. Univalent Foundations of Mathematics.* 2013.