

Improved Online Scheduling in Maximizing Throughput of Equal Length Jobs

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Abstract

Inspired by Mechanism Design, we study an online scheduling problem in maximizing throughput of equal length jobs with application in Auctions. We design monotone algorithms with improved competitive ratio for the problem in both models of restart and resume. Due to the monotonicity property, our algorithms deduce truthful mechanisms with the same competitive ratio in maximizing the total welfare of an auction problem with single-minded bidders.

1 Introduction

Inspired by Adwords of Google and Yahoo!, a phenomenally fast-growing market, auction is a motivation and a new source of important problems that attracts a lot of attention and has been extensively studying by economists, computer scientists, etc. Roughly speaking, in an auction, the auctioneer first collects bids from bidders and then decides who receive the service and how much they have to pay. The auction is designed in such a way that it yields the desired goal for the auctioneer, for example: maximizing the profit, maximizing the welfare or making the auction to be truthful. An auction is *truthful* if every bidder, while fixing the bids of other bidders, maximizes her utility by submitting her true value that she would like to pay for the service.

Auction problem Consider the auction in which at each regular period of time, the auctioneer has a non-storable *perishable* identical good that he has to sell immediately to some bidder (for example, electricity, ice-cream ... which are hardly or costly to be stored). Bidders arrive online. The presence of bidder i is only known to the auctioneer at i 's arrival time r_i . Each bidder i has a departure time d_i and has an own *private* value w_i on the allocation of p_i goods and 0 if he gets less than p_i goods before his departure. At the arrival time, bidder i announces his departure time d_i , his demand p_i and his bid b_i on the demand. The bidder may ask for p_i consecutive items or not. The utility u_i of bidder i is $w_i - v_i$ where v_i is the payment of i if her demand is satisfied, otherwise $u_i = 0$. The goal of each bidder is to maximize her own utility by choosing an appropriate bid. In this setting, a truthful mechanism is a mechanism in which every bidder maximizes her utility by reporting $b_i = w_i$. The goal of the auctioneer is to design a truthful auction which maximizes the total *welfare* – the sum $\sum_i w_i$ over all satisfied bidders.

Optimization problem The problem without the truthful property can be seen as an online-scheduling problem on a single machine. More formally, we consider the online scheduling problem where each job j is revealed at release time r_j , has processing time p_j , deadline d_j and weight w_j . A job i could be scheduled in different separated time intervals, as long as the total

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processing time is p_i . The goal is to maximize the *throughput* – the total weight of all jobs completed on time.

An algorithm is *r-competitive* if for any job sequence released by an adversary, the algorithm gain is at least r -fraction of the optimal offline solution where the whole sequence of jobs is known in advance.

Optimization versus Mechanism Design The auction problem above belongs to an important domain of mechanism design, called *single-parameter*. Intuitively, in this class, the private value of each bidder is a single parameter. The class is extensively studied in Game Theory and the characterization of truthful mechanisms in single-parameter domain is well settled.

Given the bids of all bidders $b = (b_1, \dots, b_n)$, the auction will run an *allocation* algorithm which decides who will be satisfied and a *payment* algorithm which decides how much each one has to pay. We call an allocation algorithm for a single-parameter problem *monotone* if a satisfied bidder i still wins if she increases her bid to $b'_i > b_i$.

Given the bids of all bidders $b = (b_1, \dots, b_n)$, the *critical value* v_i^c of bidder i is defined as:

$$v_i^c = \begin{cases} \min b'_i & \text{such that } i \text{ is satisfied with } b' = (b_1, \dots, b_{i-1}, b'_i, b_{i+1}, \dots, b_n), \\ \infty & \text{if no such } b'_i \text{ exists} \end{cases}$$

Intuitively, the critical value of i is the smallest possible value of bid that i 's demand will be satisfied by submitting this bid while the other bids are fixed. Truthful mechanism for single-parameter domain is well characterized by monotonicity.

Proposition 1 ([14]). *In the single-parameter domain, a deterministic mechanism is truthful if and only if its allocation algorithm is monotone and the payment algorithm collects a charge equal to the critical value from each satisfied bidder.*

This proposition is very useful in designing truthful mechanism for single-parameter problems. In this paper, we will design monotone, competitive algorithms for the optimization problem and verify that the critical payment scheme may be computed in polynomial time. That deduces truthful auction with the same competitive ratio.

1.1 Related work

We first resume previous work related to ours in aspect of optimization as well as in mechanism design.

Optimization There are two models on the problem: (i) the *preemptive model with restart*, where a job can be interrupted, but when it is scheduled again, it must be scheduled from the beginning and (ii) the *preemptive model with resume*, where in contrast, when a job is scheduled again, the previously done work can be resumed. It is known that, in both models, the deterministic competitive ratio is unbounded if jobs' processing time is unbounded [1]. Durr et al. [7] presented an algorithm that gave a tight bound $\Theta(k/\log k)$ in both models where the processing time of all jobs are bounded by k .

In the model with restart, Zheng et al. [15] provided an algorithm BAR that was 4.56-competitive for jobs with equal processing time, arbitrary weights. Chrobak et al. [5] considered randomized algorithms and they gave a tight 1.5-competitive deterministic algorithm for equal processing time, unit weight jobs.

In the model with resume, Kalyanasundaram and Pruhs [10] presented a constant competitive randomized algorithm for unit weight jobs. For the variant with arbitrary processing time

and arbitrary weight jobs, Canetti and Irani [2] showed a $\Omega(\sqrt{\log k / \log \log k})$ lower bound for any randomized competitive algorithm against an oblivious adversary. In case of equal length and arbitrary weight jobs, the best known upper bound and lower bound competitive ratio is 5 [7] and 2.59 [7, 3], respectively. For an interesting special case where jobs have unit processing time, the problem (which is also called Buffer Overflow Management) is widely studied, and the deterministic competitive ratio is between $\phi(\approx 1.618)$ [4] and 1.83 [8]. Another direction of research is to consider resource augmentation, and in [13] a deterministic online algorithm was presented which has constant competitive ratio provided that the algorithm is allowed a constant speedup of its machine compared to the adversary.

Mechanism Design Hajiaghayi et al. [9] considered an online auction problem similar to ours but bidders are interested in a single item. They showed that the greedy algorithm — the algorithm allocates every item to the bidder with highest bid who has not been satisfied yet — was monotone. As Kesselman et al. [11] showed that greedy algorithm is 2-competitive, that gave a 2-competitive truthful mechanism. Hajiaghayi et al. [9] also proved that no deterministic truthful mechanism has competitive ratio smaller than 2. Cole et al. [6] gave another truthful 2-competitive mechanism, called *prompt*, in the sense that satisfied bidders always learn their payment immediately after receiving their last item. While the processing time of jobs is bounded, a truthful auction with tight bound is deduced from the algorithm in [7].

1.2 Our contribution

In the paper, we study the problem for jobs of equal processing time and give monotone algorithms which are 4.24-competitive for both models of restart and resume in optimization problem. The algorithms can be turned into truthful mechanism using the critical payment scheme described in Proposition 1. The algorithms for both models are essentially the same and the analysis are based on charging schemes. First, we present the algorithm and illustrate the analysis for model of restart in Section 3. The charging scheme in the model of restart does not work as the adversary, in the model of resume, may schedule jobs in different pieces and it is the main difficulty in extending the algorithm for the latter model. In Section 4, we describe a more subtle charging scheme that guarantees the competitive ratio 4.24 for model of resume.

2 Preliminaries

For a job i we denote its release time by r_i , its deadline by d_i , its processing time by p_i and its weight by w_i . For all job i , $p_i = k \geq 1$ at release time r_i . All these quantities, except possibly w_i , are integers. Let C_i be the completion time of job i by an algorithm. Let $q_i(t)$ be the remaining processing time of job i for the algorithm at time t . When there is no confusion, we simply write q_i . In the model of restart, if a job i is interrupted at some time t then $q_i(t+1) = k$.

We say that an algorithm schedules a job at time t meaning that the algorithm executes an unit of this job in interval $[t, t+1)$. A job i is *pending* for the algorithm at time t if it has not been completed before and $r_i \leq t$ and $t + q_i(t) < d_i$. A job i is *urgent* at time t if $d_i < t + q_i(t) + k$.

In the model of restart, let $S_i(t)$ be the latest moment before t that an algorithm starts job i . In the model of resume, $S_i(t)$ denote the latest moment $\tau < t$ such that at time τ , the algorithm schedules job i but at the previous moment $(\tau - 1)$, the algorithm schedules other job or it is an idle-time. Again, when there is no confusion, we simply write S_i .

W.l.o.g we assume that the adversary starts a job if and only if the job will be completed and the adversary schedules job in EARLIEST DEADLINE FIRST manner. In the analysis, we

abbreviate the currently studying algorithm and the adversary by ALG and ADV, respectively.

Starting point. Roughly speaking, the main difficulty of the problem is to decide whether at a moment, executing a job with heavy weight or a more urgent job. The advantage of executing a heavy job is to guarantee the competitiveness of an algorithm if the adversary releases heavy jobs subsequently. Executing more urgent job is useful if there is no big change in the future. An online algorithm needs to balance these things.

Consider the following algorithm in model of restart. If there is no currently scheduling job, schedule the pending one with highest weight. Interrupt the currently scheduling job j if there is a new job i arriving with weight at least $2w_j$.

This algorithm is 5-competitive [12, 3] where the proof is based on a charging scheme. An idea of improvement is that if a new released job is urgent then one may delay the execution of the currently scheduling job. However, postponing that new heavy job might result in a lost of throughput if some new other heavier job is released later. Hence, an improved algorithm should be more involved in the deadlines of jobs, not only in verifying whether a job is pending. A treatment is presented in [15] in which they handled implicitly the jobs' deadlines by turning them into a function of weight. In the paper, we deal explicitly with jobs' deadlines and present new algorithms with improved bounds.

3 A 4.24-competitive Algorithm for Model of Restart.

The algorithm A Let $\beta > 1$ be a constant defined later. Initially, set $Q := \emptyset$ and $\alpha := -1$. At time t , if there is either no currently scheduling job or a job completion, then schedule the heaviest pending job. Otherwise, let j be the currently scheduling job. If there is no new released job, then continue to schedule j . Otherwise, let i be a new released job with heaviest weight. Job j is interrupted if one of the following conditions hold.

A1. **if:** $w_i \geq 2w_j$ and $w_i \geq 2^\alpha w(Q)$.

do: Schedule job i . Set $\alpha := -1$ and $Q = \emptyset$.

A2. **if:** $\alpha = -1$, $\beta w_j \leq w_i \leq 2w_j$, i is urgent and j can be scheduled later, i.e. $d_j \geq t + 2k$.

do: Schedule job i^* which is the heaviest among all jobs with deadline smaller than $t + 2k$, i.e., $i^* = \arg \max\{w_\ell : d_\ell \leq t + 2k\}$. Set $\alpha = 1$ and $Q := \{j\}$.

A3. **if:** i is urgent, $w_i \geq 2w_j + w_{j-1}$ where $(j - 1)$ is the job previously preempted by j and there is no job ℓ satisfying the following conditions:

$$S_j(t) + 2k \leq d_\ell < t + 2k, \quad w_\ell \geq w_j.$$

do: Schedule job i .

At any interruption, if $\alpha \geq 0$ then $\alpha := \alpha + 1$. If there is a job completion then set $\alpha = -1$ and $Q = \emptyset$. Note that, if an interruption is satisfied both conditions [A1] and [A3] then we refer that interruption to [A1].

We explain the intuition of the algorithm. Informally, the algorithm maintains a counter α indicating whether there exists an interruption by condition [A2] which is not followed by an interruption of type [A1]. If there exists, α also indicates the number of job interruptions from the last job q interrupted by condition [A2] and the set Q stores that job. By condition [A2] (on α), the algorithm does not allow two interruptions of type [A2] without an interruption of type

[A1] in between. Condition [A3] means that an urgent job j may be interrupted by another urgent job i if the new job i is heavy enough and no pending job could be scheduled later to compensate the weight of i in case i is not scheduled.

Observation 1. 1. An interruption of type [A2] is preceded by a job completion or an interruption of type [A1].

2. The first interruption of type [A3] in a phase is preceded by an interruption of type [A2]. Moreover, there are at most 2 consecutive interruptions of type [A3].

Proof. 1. By contradiction, suppose that i_1 is interrupted by i_2 and i_2 is interrupted according to condition [A2] by i_3 . Then the interruption of i_1 by i_2 is not of type [A2] since otherwise counter $\alpha \geq 1$ and i_3 cannot interrupt i_2 according to condition [A2]. Moreover, if i_2 interrupts i_1 according to [A3], meaning i_2 is urgent then i_3 cannot interrupt i_2 according to condition [A2] because i_2 may not be scheduled later. Therefore, the interruption of i_1 is of type [A1].

2. The first claim is obvious from the definition of the algorithm. We prove the second claim by contradiction. Suppose that there are 4 jobs i_4, i_3, i_2, i_1, i_0 such that i_3 interrupt i_4 according to [A2] and i_2, i_1, i_0 interrupts i_3, i_2, i_0 respectively according to [A3]. We claim that in fact, i_0 is heavy enough to preempt i_1 according to [A1]. We have:

$$w_{i_0} \geq 2w_{i_1} + w_{i_2} \geq 2(2w_{i_2} + w_{i_3}) + w_{i_2} \geq 5(2w_{i_3} + w_{i_4}) + 2w_{i_3} > 2^4 w_{i_4}$$

where the inequalities are due to condition of [A3] and $w_{i_3} > w_{i_4}$. Hence, i_0 interrupts i_1 according to [A1]. □

Analysis We analyze the competitiveness of the algorithm by a charging scheme. We divide the schedule of the algorithm into *phases* where each phase is an interval defined by two consecutive job-completion times (except the first phase defined by time 0 and the first job completion time). We say that a *phase of job i* if i is completed in this phase by ALG. Consider a phase of job i , let $f(i)$ be the first job started by ALG in this phase. Remark that, $f(i)$ is not necessarily completed by ALG. Until the end of the section, S_j means $S_j(C_i)$ for all job j . First, we claim the following lemma.

Lemma 1. Consider a sequence of jobs $m, \dots, 1, 0$ where job ℓ interrupts job $(\ell + 1)$ for $1 \leq \ell \leq m - 1$ and job 0 interrupts job 1 by condition [A1]. Then

1. $w_\ell \leq w_0 \cdot 2^{-\ell}$ for $0 \leq \ell \leq m$.
2. $w_{\ell'} \leq w_0 \cdot 2^{-\ell}$ where ℓ' be a job started by ADV in $[S_{\ell+1}, S_\ell)$ and has not been completed by ALG. Consequently, the total weight of jobs which have not been completed by ALG and are started by ADV from the beginning of the phase to S_0 is bounded by $2w_0 - 2w_m$.

Proof. 1. We prove the claim by induction. It is straightforward for $\ell = 0$. Suppose that $w_\ell \leq w_0 \cdot 2^{-\ell}$. If job ℓ interrupts job $(\ell + 1)$ by condition [A1] or [A3] then $w_{\ell+1} \leq w_\ell \cdot 2^{-1} \leq w_0 \cdot 2^{-\ell+1}$. Otherwise, let h ($h < \ell$) be the job that interrupts job $(h + 1)$ by condition [A1]. This interruption exists since the assumption of job 0. By definition of [A1], $w_\ell \leq w_h \cdot 2^{h-\ell}$. Moreover, by induction hypothesis, $w_h \leq w_0 \cdot 2^{-h}$. Therefore, $w_\ell \leq w_0 \cdot 2^{-\ell}$.

2. By contradiction, let ℓ be the largest index such that there exists a job ℓ' started by ADV in $[S_{\ell+1}, S_\ell)$ and $w_{\ell'} > w_0 \cdot 2^{-\ell}$. Hence, for any job $h > \ell$, $w_h \leq w_0 \cdot 2^{-h} < w_{\ell'} \cdot 2^{\ell-h}$. As ℓ' has not been completed by ALG, ℓ' may interrupt job $(\ell+1)$ by condition [A1]. Hence, job ℓ' is the same as job ℓ . This contradicts to the fact that $w_\ell \leq w_0 \cdot 2^{-\ell} < w_{\ell'}$.

The total weighs of jobs which have not been completed by ALG and are started by ADV before S_0 in the phase is bounded by $\sum_{\ell=0}^{m-1} w_0 \cdot 2^\ell \leq 2w_0 - 2^{m-1} \leq 2w_0 - 2w_m$. □

The Charging Scheme

1. If a job i scheduled by ADV and i has been completed by ALG before then charge w_i to job i (self-charge).
2. If a job j is started by ADV in the phase of job i and j has not been completed before by ALG, then charge w_j to i .
3. For each phase of job i , if there exists a next phase of job j such that $S_{f(j)} = C_i$ then charge $2w_{f(j)}$ from job i to job j .

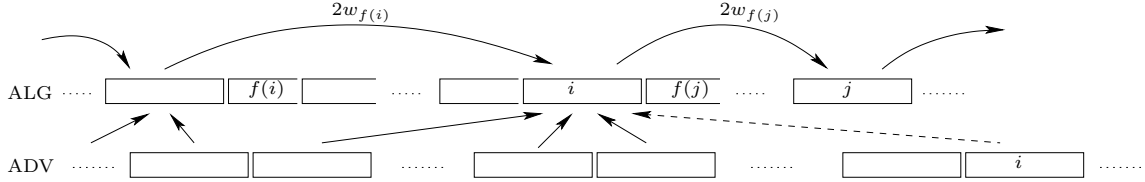


Figure 1: Illustration of the charging scheme. The dashed pointer and the curly pointer represent the self-charge and the charge of step 3.

Consider a phase of job i and let i' be the job started by the ALG just after finished i . Note that in case there exists no such job i' , conventionally $w_{i'} = 0$. In the analysis, we will argue that the total charge that i receives after step 2 of the charging scheme is at most $r \cdot w_i - 2w_{f(i)} + 2w_{i'}$ where r is a constant that is revealed later. Therefore, each job i completed by ALG receives at most $r \cdot w_i$ in the end of the charging scheme, which deduces the competitive ratio r of the algorithm. Until the end of the section, without state differently, we refer the charge of a job is the one that the job receives after step 2 of the charging scheme.

We assume without loss of generality that if ADV starts a job, it will complete the job. We say that a phase is of *type 1, 2, 3 or 0* if the last interruption in the phase is according to [A1], [A2], [A3] or there is no interruption in the phase, respectively. If a phase of job i is 0-type then i receives at most one charge from a job j started in $[S_i, C_i)$ and probably one self-charge. As j does not interrupt i and j is not completed before, $w_j < 2w_i$. Hence, the total charge that i receives is at most $3w_i$. In the following, we bound the charge when the job i 's phase is of type 1, 2 and 3. Denote $t = f(i)$ and i' be the job started by the ALG just after finished i (at time C_i).

Lemma 2. *If the phase of job i is of type 1 then the total charge that i receives is at most $\max\{3 + \beta, 4\}w_i - 2w_t + 2w_{i'}$.*

Proof. By Lemma 1, the total charge that i receives from jobs started before S_i in ADV is at most $2w_i - 2w_t$. Consider the job j scheduled by ADV in $[S_i, C_i)$. We have $w_j < 2w_i$ since otherwise, j can interrupt i by [A1]. If $w_j \leq \beta w_i$ or j has been completed by ALG or i receives no self-charge then the total charge that i receives, including the self-charge and the charge

from j , is at most $\max\{2w_i - 2w_t + w_i + \beta w_i, 2w_i - 2w_t + w_i, 2w_i - 2w_t + w_j\} \leq (3 + \beta)w_i - 2w_t$. The remaining case is that $\beta w_i < w_j < 2w_i$ and i receives a self-charge. Let τ be the moment that ADV start j .

1. j is not urgent at τ

As j is still pending at completion time of i , $w_{i'} \geq w_j > w_i$. Hence, the total charge that i receives in this case is $2w_i - 2w_t + w_i + w_j \leq 4w_i - 2w_t + w_{i'}$.

2. j is urgent at τ

Since i receives a self-charge, $d_i \geq \tau + 2k$. As j is urgent and $w_j > \beta w_i$, i would have been preempted according to condition [A2], contradicts to the fact that the phase is of type 1. □

Lemma 3. *If the phase of job i is of type 2 then the total charge that i receives is at most $\max\left\{\frac{6}{\beta} - 1, 4\right\} \cdot w_i - 2w_t + 2w_{i'}$.*

Proof. Let i_1 be the job interrupted by i . By observation, either i_1 is preceded by a job completion or an interruption of type [A1]. Then, by Lemma 1, the total charge that i receives from jobs started by ADV before S_{i_1} is at most $2w_{i_1} - 2w_t$ (in case of job completion before S_{i_1} , that charge is 0, also bounded by $2w_{i_1} - 2w_t$). Due to the fact that all jobs have the same length, there are at most two jobs j_1 and j started by ADV in intervals $[S_{i_1}, S_i]$ and $[S_i, C_i]$, respectively (Figure 2). As i interrupts i_1 by condition [A2], at time C_i job i_1 is still pending, so at that moment, ALG will schedule some job i' with weight at least w_{i_1} . Moreover, $d_i < S_i + 2k$ so either i receives no self-charge or $j = i$. Hence, the total charge that i receives is at most $2w_{i_1} - 2w_t + w_{j_1} + w_j$. In the following, we bound the term $2w_{i_1} + w_{j_1} + w_j$. We have that $w_{j_1} < 2w_{i_1}$ since otherwise, job j_1 would interrupt job i_1 by condition [A1]. Let τ be the moment that ADV starts j .

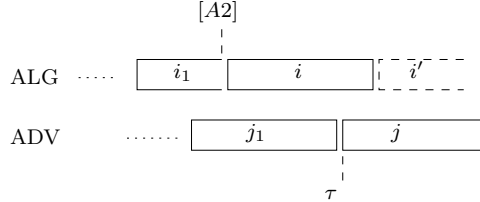


Figure 2: Illustration for a phase of type 2

1. j is not urgent at τ

As j is still pending at completion time of i , $w_{i'} \geq w_j > w_i$. Hence, $2w_{i_1} + w_{j_1} + w_j \leq 4w_i + 2w_{i'}$ since $w_{j_1} < 2w_{i_1} \leq 2w_i$ and $w_j \leq w_{i'}$.

2. j is urgent at τ

(a) **There exists job ℓ such that $S_i + 2k \leq d_\ell < \tau + 2k$ and $w_\ell \geq w_i$.**

In this case, we have $w_{i'} \geq w_\ell \geq w_i$. If $w_{j_1} \leq w_i$ then $w_{i'} \geq w_{j_1}$. If $w_{j_1} > w_i$ then j_1 is not urgent at its starting time by the ADV since otherwise, j_1 would have

interrupted i_1 according to [A2]. Hence, j_1 is pending at S_i . By the job selection in condition [A2], j_1 is also pending at time C_i . Thus, $w_{i'} \geq w_{j_1}$.

$$2w_{i_1} + w_{j_1} + w_j \leq 6w_{i_1} + w_{i'} \leq \frac{6}{\beta}w_i + w_{i'} \leq \left(\frac{6}{\beta} - 1\right) \cdot w_i + 2w_{i'}$$

where the first is due to $w_j \leq 4w_{i_1}$ (otherwise, j would have interrupted i by [A1])

(b) **There does not exist job ℓ such that $S_i + 2k \leq d_\ell < \tau + 2k$ and $w_\ell \geq w_i$.**

As j is urgent, $w_j < 2w_i + w_{i_1}$ since otherwise j would have preempted job i by [A3]. Hence,

$$2w_{i_1} + w_{j_1} + w_j < 2w_{i_1} + w_{j_1} + 2w_i + w_{i_1}$$

If $d_{j_1} \geq S_i + 2k$ then $w_{i'} \geq \max\{w_{i_1}, w_{j_1}\}$, so $2w_{i_1} + w_{j_1} + 2w_i + w_{i_1} < 4w_i + 2w_{i'}$ (note that $w_i > w_{i_1}$). If $d_{j_1} < S_i + 2k$ and as j_1 is not completed before by ALG then by the choice of scheduled job at interruption [A2], $w_{j_1} \leq w_i$. Therefore, we have $2w_{i_1} + w_{j_1} + 2w_i + w_{i_1} < 4w_i + 2w_{i'}$.

□

Lemma 4. *If the phase of job i is of type 3 then the total charge that i receives is at most $\max\left\{\frac{3\beta+11}{2\beta+1}, 4\right\} w_i - 2w_t + 2w_{i'}$.*

Proof. By observation, there are at most two consecutive interruptions [A3]. We consider first the case with one interruption. The case of two interruptions is analyzed similarly.

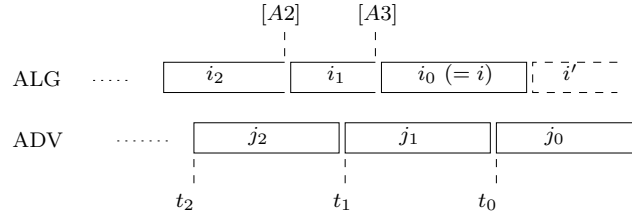


Figure 3: Illustration for the phase with one interruption [A3]

1. There is only one interruption of type [A3].

Let $i_0(= i), i_1, i_2$ be jobs started by ALG in the phase such that i_0, i_1 interrupt i_1, i_2 according to [A3] and [A2], respectively. Let j_0, j_1 and j_2 be jobs started by ADV in $[S_{i_0}, C_{i_0}), [S_{i_1}, S_{i_0})$ and $[S_{i_2}, S_{i_1})$, respectively (Figure 3). By Lemma 1, the total charge of jobs started by ADV from the beginning of the phase to S_{i_2} is at most $2w_{i_2} - 2w_t$ (in case of job completion before S_{i_2} , that charge is 0, also bounded by $2w_{i_2} - 2w_t$). As job i is new released and urgent at time S_i , i receives no self-charge. In the following, we argue that $w_{j_2} + w_{j_1} + w_{j_0} + 2w_{i_2} \leq \max\left\{\frac{3\beta+11}{2\beta+1}, 4\right\} w_{i_0} + 2w_{i'}$

(a) **Case $w_{j_1} > w_{i_0}$.** Then by condition [A3] (at time $t = S_{i_0}$), we have either $d_{j_1} \geq S_{i_0} + 2k$ or $d_{j_1} < S_{i_1} + 2k$. In the former, $w_{i'} \geq w_{j_1}$. Hence,

$$4w_{i_0} + 2w_{i'} > 4(2w_{i_1} + w_{i_2}) + w_{j_1} \geq 8w_{i_1} + 4w_{i_2} + w_{j_1}$$

However, as j_2, j_0 do not interrupt i_2, i_0 according to [A1] so $w_{j_2} \leq 2w_{i_2}$ and $w_{j_0} \leq 8w_{i_2}$. Therefore, $4w_{i_0} + 2w_{i'} > w_{j_0} + w_{j_1} + w_{j_2} + 2w_{i_2}$.

In the latter, job j_1 is urgent at its starting time t_1 by the ADV. Moreover, we know that no job ℓ satisfying $S_{i_1} + 2k \leq d_\ell < S_{i_0} + 2k$ and $w_\ell \geq w_{i_1}$ by definition of [A3]. So there is no job ℓ satisfying $S_{i_1} + 2k \leq d_\ell < t_1 + 2k$ and $w_\ell \geq w_{i_1}$. As $w_{j_1} > w_{i_0}$, j_1 would have interrupted i_1 according to condition [A3] (contradiction).

- (b) **Case** $w_{j_1} \leq w_{i_0}$ **and** $w_{j_2} > w_{i_1}$. Then j_2 is not urgent at its starting time t_2 by the ADV. By [A2], $d_{j_2} \geq S_{i_1} + 2k$. Furthermore, by definition of condition [A3], we have $d_{j_2} \geq S_{i_0} + 2k$. Hence, $w_{i'} \geq w_{j_2} > w_{i_1} > w_{i_2}$. Therefore,

$$\begin{aligned} 4w_{i_0} + 2w_{i'} &> w_{j_1} + 3w_{i_0} + w_{j_2} + w_{i_2} \geq w_{j_1} + 3(2w_{i_1} + w_{i_2}) + w_{j_2} + w_{i_2} \\ &\geq w_{j_1} + 8w_{i_2} + w_{j_2} + 2w_{i_2} \geq w_{j_1} + w_{j_0} + w_{j_2} + 2w_{i_2} \end{aligned}$$

- (c) **Case** $w_{j_1} \leq w_{i_0}$ **and** $w_{j_2} \leq w_{i_1}$. We have:

$$\begin{aligned} w_{j_1} + w_{j_2} + w_{j_0} + 2w_{i_2} &\leq w_{i_0} + w_{i_1} + 8w_{i_2} + 2w_{i_2} \\ &\leq w_{i_0} + (w_{i_1} + \frac{1}{2}w_{i_2}) + \frac{19}{2(2\beta + 1)}(2\beta + 1)w_{i_2} \\ &\leq w_{i_0} + \frac{w_{i_0}}{2} + \frac{19}{2(2\beta + 1)}w_{i_0} = \frac{3\beta + 11}{2\beta + 1} \cdot w_{i_0} \end{aligned}$$

where we use the facts that $w_{j_0} \leq 8w_{i_2}$ and $(2\beta + 1)w_{i_2} \leq w_{i_0}$.

2. **There are two interruptions of type [A3].** This analysis is similar as previous (see Appendix). □

Theorem 1. *The algorithm is $(2 + \sqrt{5})$ (≈ 4.24)-competitive when $\beta = \sqrt{5} - 1$.*

Proof. Due to previous lemmas, the charge that a job i completed by ALG is at most $r = \max\{4, 3 + \beta, \frac{6}{\beta} - 1, \frac{3\beta + 11}{2\beta + 1}\}$. The ratio r attains minimum value $(2 + \sqrt{5})$ when $\beta = \sqrt{5} - 1$. □

Corollary 1. *There exists a 4.24-competitive truthful mechanism for the auction problem in which bidders ask for consecutive items.*

Proof. Consider Algorithm A with different terminologies, bidder instead of job and bid instead of weight. By the algorithm, a satisfied bidder will also be satisfied if she increases the bid while the other stay the same. Hence, the algorithm is monotone. The critical bid-values can be computed in polynomial time in the number of bidders in the auction. By Proposition 1, there exists a 4.24-competitive truthful mechanism for the auction problem in which bidders ask for consecutive items. □

4 A 4.24-competitive Algorithm for Model of Resume.

The algorithm is essentially the same but now some conditions concern to the remaining processing time of jobs rather than their entire processing time.

The algorithm B Initially, set $Q := \emptyset$ and $\alpha := -1$. At time t , if there is either no currently scheduling job or a job completion, then schedule the heaviest job. Otherwise, let j be the currently scheduling job. If there is no new released job, then continue to schedule j . Otherwise, let i be a new released job with heaviest weight. Job j is interrupted if one of the following conditions hold.

B1. if: $w_i \geq 2w_j$ and $w_i \geq 2^\alpha w(Q)$.

do: Schedule job i . Set $\alpha := -1$ and $Q = \emptyset$.

B2. if: $\alpha = -1$, $\beta w_j \leq w_i \leq 2w_j$, i is urgent and j can be scheduled later, i.e. $d_j \geq t + k + q_j$,

do: Schedule job i^* which is the heaviest among all jobs with deadline smaller than $t + 2k$, i.e., $i^* = \arg \max\{w_\ell : d_\ell \leq t + 2k\}$. Set $\alpha = 1$ and $Q := \{j\}$.

B3. if: i is urgent, $w_i \geq 2w_j + w_{j-1}$ where $(j-1)$ is the job previously preempted by j and there is no another job ℓ satisfying the following conditions:

$$S_j + k + q_\ell(t) \leq d_\ell < t + k + q_\ell(t), \quad w_\ell \geq w_j.$$

do: Schedule job i .

At any interruption, if $\alpha \geq 0$ then $\alpha := \alpha + 1$. If there is a job completion then set $\alpha = -1$ and $Q = \emptyset$. Note that, if an interruption is satisfied both conditions [B1] and [B3] then we refer that interruption to [B1].

Analysis The charging scheme in the model of restart does not work as now the adversary may schedule jobs in different pieces and it is the main difficulty in extending the algorithm for the model of resume. Using the equal-length property of jobs, we analyze the algorithm by a different charging scheme. The new charging scheme, which has been used in [7], is more subtle.

The Charging Scheme For convenience we renumber the jobs completed by the algorithm from 1 to n , such that the completion times are ordered $C_1 < \dots < C_n$. Also we denote $C_0 = 0$. For every $i = 1, \dots, n$ we divide $[C_{i-1}, C_i)$ further into intervals: Let $a = \lceil (C_i - C_{i-1})/k \rceil$. The first interval is $[C_{i-1}, C_i - (a-1)k)$. The remaining intervals are $[C_i - (b+1)k, C_i - bk)$ for every $b = a-2, \dots, 0$. We label every interval I with a pair (b, i) such that $I = [s, C_i - bk)$ for $s = \max\{C_{i-1}, C_i - (b+1)k\}$.

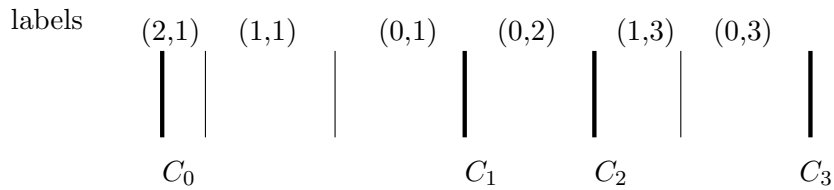


Figure 4: The intervals as used by the charging procedure.

The charging will be done by the following procedure, which maintains for every interval $[s, t)$ a set of jobs P that are started before t by the adversary and that are not yet charged to some job of the algorithm.

Initially $P = \emptyset$.

For all intervals $[s, t)$ as defined above in left to right order, do

- Let (b, i) be the label of the interval.
- Add to P all jobs j started by the adversary in $[s, t)$.
- If P is not empty, then among all jobs started in $[s, t)$ (jobs have just been added to P), let j be the job with smallest deadline. If no such job exists, let j be the smallest deadline job in P . Remove j from P and charge it to i . Mark $[s, t)$ with j .
- If P is empty, then $[s, t)$ is not marked.
- Denote by P_t the current content of P .

Note that, no job i receives charge from job started after C_i . Using the fact that all jobs have the same length, we prove the following main lemma.

Lemma 5. *For every interval $[s, t)$, all jobs $j \in P_t$ are still pending for the algorithm at time t .*

Proof. Assume that P_t is not empty, and let j be the job in P_t with the smallest deadline.

First we claim that there is a time s_0 , such that every interval contained in $[s_0, t)$ is marked with some job j' satisfying either (1) $s_0 \leq r_{j'}$ and $d_{j'} \leq d_j$; or (2) job j' is started by ADV in that interval.

The existence of s_0 is shown by a kind of pointer chasing: Let $[s', t')$ be the interval where the adversary started j . So j entered P by the charging procedure at this interval. Job j was in P during all the iterations until $[s, t)$, so every interval between t' and t is marked with some job of deadline at most d_j or is marked with some job started by ADV in the interval. Let \mathcal{M} be the set of these jobs. If for every $j' \in \mathcal{M}$ we have $s' \leq r_{j'}$, we choose $s_0 = s'$ and we are done. Otherwise let $j' \in \mathcal{M}$ be the job with smallest release time. So $r_{j'} < s'$. Let $[s'', t'')$ be the interval where the adversary started j' . By the same argument as above, during the iteration over the intervals between s'' and s' , job j' was in P . Therefore every such interval was marked with some job with deadline at most $d_{j'} \leq d_j$ or was marked with some job started by ADV in the interval. Now we repeat for s'' the argument we had for s' . Eventually we obtain a valid s_0 , since P was initially empty.

Now let \mathcal{M} be the set of jobs charged during all intervals in $[s_0, t)$. In an EARLIEST DEADLINE FIRST schedule of the adversary, job j would complete not before $s_0 + (|\mathcal{M}| + 1)k$. But any interval has size at most k , so $t - s_0 \leq |\mathcal{M}|k$. We conclude that $d_j \geq t + k$, which shows that j is still pending for the algorithm at time t . \square

Using Lemma 5, the remaining analysis follows the same structure as in the model of restart, though with different details in presented in the appendix.

Theorem 2. *The algorithm B is $(2 + \sqrt{5})(\approx 4.24)$ -competitive when $\beta = \sqrt{5} - 1$.*

Corollary 2. *There exists a 4.24-competitive truthful mechanism for the auction problem in which bidders ask for separate items.*

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Appendix

A Remaining proof of Lemma 4

There are two interruptions of type [A3]

Let $i_0(=i), i_1, i_2, i_3$ be jobs started by ALG in the phase such that i_0, i_1, i_2 interrupt i_1, i_2, i_3 according to [A3], [A3] and [A2], respectively. Let j_0, j_1, j_2 and j_3 be jobs started by ADV in $[S_{i_0}, C_{i_0}), [S_{i_1}, S_{i_0}), [S_{i_2}, S_{i_1})$ and $[S_{i_3}, S_{i_2})$, respectively (Figure 5). By Lemma 1, the total charge of jobs started by ADV from the beginning of the phase to S_{i_3} is at most $2w_{i_3} - 2w_t$. As job i is new released and urgent at time S_i , i receives no self-charge. In the following, we argue that $w_{j_3} + w_{j_2} + w_{j_1} + w_{j_0} + 2w_{i_3} \leq \max\left\{\frac{3\beta+11}{2\beta+1}, 4\right\} w_{i_0} + 2w_{i'}$. The cases will be proceeded similarly as previous.

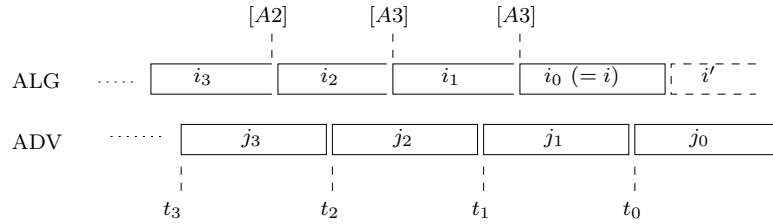


Figure 5: Illustration for the phase with two interruptions [A3]

1. **Case $w_{j_1} > w_{i_0}$.** Then by condition [A3], we have either $d_{j_1} \geq S_{i_0} + 2k$ or $d_{j_1} < S_{i_1} + 2k$. In the former, $w_{i'} \geq w_{j_1}$. Hence,

$$4w_{i_0} + 2w_{i'} > 4(2w_{i_1} + w_{i_2}) + w_{j_1} \geq 8w_{i_1} + 4w_{i_2} + w_{j_1} \geq 20w_{i_2} + 8w_{i_3} + w_{j_1}$$

However, as j_3, j_2, j_0 do not interrupt i_3, i_2, i_0 according to [A1] so $w_{j_2} \leq 2w_{i_2}, w_{j_2} \leq 4w_{i_2}$ and $w_{j_0} \leq 16w_{i_2}$. Therefore, $4w_{i_0} + 2w_{i'} > w_{j_0} + w_{j_1} + w_{j_2} + w_{j_3} + 2w_{i_3}$.

In the latter, job j_1 is urgent at its starting time t_1 by the ADV. Moreover, we know that no job ℓ satisfying $S_{i_1} + 2k \leq d_\ell < S_{i_0} + 2k$ and $w_\ell \geq w_{i_1}$ by definition of [A3]. So there is no job ℓ satisfying $S_{i_1} + 2k \leq d_\ell < t_1 + 2k$ and $w_\ell \geq w_{i_1}$. As $w_{j_1} > w_{i_0}$, j_1 would have interrupted i_1 according to condition [A3] (contradiction).

2. **Case $w_{j_1} \leq w_{i_0}$ and $w_{j_2} > w_{i_1}$.** Then j_2 is not urgent at its starting time t_2 by the ADV since otherwise, j_2 would have preempted i_2 according to [A3]. By [A2], $d_{j_2} \geq S_{i_1} + 2k$. Furthermore, by definition of condition [A3], we have $d_{j_2} \geq S_{i_0} + 2k$. Hence, $w_{i'} \geq w_{j_2} > w_{i_1}$. Therefore,

$$\begin{aligned} 4w_{i_0} + 2w_{i'} &> w_{j_1} + 3w_{i_0} + w_{j_2} + w_{i_1} \geq w_{j_1} + w_{j_2} + 3(2w_{i_1} + w_{i_2}) + 2w_{i_2} + w_{i_3} \\ &> w_{j_1} + w_{j_2} + 6(2w_{i_2} + w_{i_3}) + 6w_{i_3} \\ &> w_{j_1} + w_{j_2} + 16w_{i_3} + 2w_{i_3} + 2w_{i_3} \\ &> w_{j_1} + w_{j_2} + w_{j_0} + w_{j_3} + 2w_{i_3} \end{aligned}$$

So the total charge that i receives is smaller than $4w_{i_0} + 2w_{i'} - 2w_t$.

3. **Case** $w_{j_1} \leq w_{i_0}$ **and** $w_{j_2} \leq w_{i_1}$. We have:

$$\begin{aligned}
w_{j_1} + w_{j_2} + w_{j_0} + w_{j_3} + 2w_{i_3} &\leq w_{i_0} + w_{i_1} + 16w_{i_3} + 2w_{i_3} + 2w_{i_3} \\
&\leq w_{i_0} + w_{i_1} + \frac{40 - \beta}{5\beta + 2}w_{i_3} + \frac{101}{5\beta + 2}w_{i_2} \\
&\leq w_{i_0} + w_{i_1} + \frac{40 - \beta}{5\beta + 2}(2w_{i_2} + w_{i_3}) + \frac{21 + 2\beta}{5\beta + 2}w_{i_2} \\
&= w_{i_0} + \frac{4\beta + 42}{5\beta + 3}w_{i_1} + \frac{21 + 2\beta}{3\beta + 1}w_{i_2} \\
&\leq w_{i_0} + \frac{2\beta + 21}{5\beta + 3}(2w_{i_1} + w_{i_2}) \leq \frac{7\beta + 24}{5\beta + 3}w_{i_0} < \frac{3\beta + 11}{2\beta + 1}w_{i_0}
\end{aligned}$$

where the third inequality is due to $w_{i_2} \geq \beta w_{i_3}$.

B Lemmas in Models of Resume

The following lemma is of the same spirit as Lemma 1.

Lemma 6. *Consider a phase in which job ℓ interrupts job $(\ell + 1)$ for $1 \leq \ell \leq m - 1$ and job 0 interrupts job 1 by condition [B1]. Then*

1. $w_\ell \leq w_0 \cdot 2^{-\ell}$ for $0 \leq \ell \leq m$.
2. $w_{\ell'} \leq w_0 \cdot 2^{-\ell'}$ for any job ℓ' in P_{S_ℓ} . Consequently, the total charge from the beginning of the phase to S_0 is bounded by $2w_0 - 2w_m$.

Proof. Using Lemma 5, the proof is similar to the one of Lemma 1. □

Denote $t = f(i)$ and i' be the job started by the ALG just after finished i (at time C_i).

Lemma 7. *If the phase of job i is of type 1 then the total charge that i receives is at most $\max\{3 + \beta, 4\}w_i - 2w_t + 2w_{i'}$.*

Proof. By Lemma 6, the total charge that i receives from jobs started before S_i in ADV is at most $2w_i - 2w_t$. Let j be a marked job in $[S_i, C_i)$. We have $w_j < 2w_i$ since otherwise, j can interrupt i by [B1]. If $w_j \leq \beta w_i$ or j has been completed by ALG or i receives no self-charge then the total charge that i receives, including the self-charge and the charge from j , is at most $\max\{2w_i - 2w_t + w_i + w_j, 2w_i - 2w_t + w_i, 2w_i - 2w_t + w_j\} \leq (3 + \beta)w_i - 2w_t$. The remaining case is that $\beta w_i < w_j < 2w_i$ and i receives a self-charge. Let τ be the moment that ADV starts j . If $\tau < S_i = S_i(C_i)$ then job j , which is pending at time S_i (by Lemma 5), would be chosen instead of job i since $w_j > w_i$. Hence, $\tau \geq S_i$.

1. j is not urgent at τ

As j is still pending at completion time of i , $w_{i'} \geq w_j > w_i$. Hence, the total charge that i receives in this case is $2w_i - 2w_t + w_i + w_j \leq 4w_i - 2w_t + w_{i'}$.

2. j is urgent at τ

Since i receives a self-charge, $d_i \geq \tau + k + q_i$, where k means the ADV schedules at least k units of job j after time τ , i.e., i can be scheduled later. As j is urgent and $w_j > \beta w_i$, i would have been preempted according by j to condition [B2], contradicts to the fact that the phase is of type 1.

□

Lemma 8. *If the phase of job i is of type 2 then the total charge that i receives is at most $\max\left\{\frac{6}{\beta} - 1, 4\right\} \cdot w_i - 2w_t + 2w_{i'}$.*

Proof. Let i_1 be the job interrupted by i . By observation, either i_1 is preceded by a job completion or an interruption of type [B1]. Then, by Lemma 1, the total charge that i receives from jobs started by ADV before S_{i_1} is at most $2w_{i_1} - 2w_t$ (in case of job completion before S_{i_1} , that charge is 0, also bounded by $2w_{i_1} - 2w_t$). Let j_1 and j be marked jobs of intervals $[S_{i_1}, S_i)$ and $[S_i, C_i)$, respectively. As i interrupts i_1 by condition [B2], at time C_i job i_1 is still pending, so at that moment, ALG will schedule some job i' such that $w_{i'} \geq w_{i_1}$.

We claim that the total charge that i receives is at most $2w_{i_1} - 2w_t + w_{j_1} + w_j$ by arguing either i receives no self-charge or $i = j$ or $i = j_1$. As i is scheduled according to [B2], $d_i < S_i + k + q_{i_1}(S_i)$. By the job selection in [B2], i is either a new released job, i.e., $r_i = S_i$ or i is a job released at time later than S_{i_1} , i.e., $r_i > S_{i_1}$ since otherwise, i would have been scheduled instead of i_1 at time S_{i_1} because $w_i > w_{i_1}$. In the former, if i receives a self-charge and $j \neq i$ then $d_i \geq S_i + 2k$ (contradiction). In the later, as in time interval $[S_{i_1}, S_i)$, the algorithm schedules only job i_1 so $q_i(S_i) = k$. So, again, if i receives a self-charge and $i \neq j, i \neq j_1$ then by the procedure of charging scheme, job j_1 is started in $[S_{i_1}, S_i)$. Hence, $d_i \geq S_{i_1} + 3k \geq S_i + 2k$ (contradiction).

In the following, we bound the term $2w_{i_1} + w_{j_1} + w_j$. We have that $w_{j_1} < 2w_{i_1}$ since otherwise, job j_1 would interrupt job i_1 by condition [B1]. If $w_j \leq w_i$ then the term is straightforwardly bounded by $4w_{i_1} + 2w_{i'}$. Let τ be the moment that ADV starts j . If j is not urgent at τ then $w_{i'} \geq w_j$, so $2w_{i_1} + w_{j_1} + w_j < 4w_{i_1} + 2w_{i'}$. In the remaining, we are interested in case $w_j > w_i$ and j is urgent at time τ . Similarly as previous lemma, $\tau > S_i$.

1. **There exists job ℓ such that $S_i + k + q_\ell \leq d_\ell < \tau + k + q_\ell$ and $w_\ell \geq w_i$.**

In this case, we have $w_{i'} \geq w_\ell \geq w_i$. If $w_{j_1} \leq w_i$ then $w_{i'} \geq w_{j_1}$. If $w_{j_1} > w_i$ then j_1 is not urgent at its starting time by the ADV since otherwise, j_1 would have interrupted i_1 according to [B2]. Hence, j_1 is pending at S_i . By the job selection in condition [B2], j_1 is also pending at time C_i . Thus, $w_{i'} \geq w_{j_1}$.

$$2w_{i_1} + w_{j_1} + w_j \leq 6w_{i_1} + w_{i'} \leq \frac{6}{\beta}w_i + w_{i'} \leq \left(\frac{6}{\beta} - 1\right) \cdot w_i + 2w_{i'}$$

where the first is due to $w_j \leq 4w_{i_1}$ (otherwise, j would have interrupted i by [B1])

2. **There does not exist job ℓ such that $S_i + k + q_\ell \leq d_\ell < \tau + k + q_\ell$ and $w_\ell \geq w_i$.**

As j is urgent, $w_j < 2w_i + w_{i_1}$ since otherwise j would have preempted job i by [B3]. Hence,

$$2w_{i_1} + w_{j_1} + w_j < 2w_{i_1} + w_{j_1} + 2w_i + w_{i_1}$$

If $d_{j_1} \geq S_i + 2k$ then $w_{i'} \geq \max\{w_{i_1}, w_{j_1}\}$, so $2w_{i_1} + w_{j_1} + 2w_i + w_{i_1} < 4w_i + 2w_{i'}$ (note that $w_i > w_{i_1}$). If $d_{j_1} < S_i + 2k$ and as j_1 is not completed before by ALG then by the choice of scheduled job at interruption [B2], $w_{j_1} \leq w_i$. Therefore, we have $2w_{i_1} + w_{j_1} + 2w_i + w_{i_1} < 4w_i + 2w_{i'}$.

□

Lemma 9. *If the phase of job i is of type 3 then the total charge that i receives is at most $\max\left\{\frac{3\beta+11}{2\beta+1}, 4\right\} w_i - 2w_t + 2w_{i'}$.*

Proof. By observation, there are at most two consecutive interruptions of type [B3]. We consider the cases with one interruption or interruptions of type [B3].

1. There is only one interruption of type [B3].

Let $i_0(= i), i_1, i_2$ be jobs started by ALG in the phase such that i_0, i_1 interrupt i_1, i_2 according to [B3] and [B2], respectively. Let j_0, j_1 and j_2 be marked jobs of intervals $[S_{i_0}, C_{i_0}), [S_{i_1}, S_{i_0})$ and $[S_{i_2}, S_{i_1})$, respectively. By Lemma 1, the total charge from the beginning of the phase to S_{i_2} is at most $2w_{i_2} - 2w_t$ (in case of job completion before S_{i_2} , that charge is 0, also bounded by $2w_{i_2} - 2w_t$). As job i is new released and urgent at time S_i , i receives no self-charge. In the following, we argue that $w_{j_2} + w_{j_1} + w_{j_0} + 2w_{i_2} \leq \max\left\{\frac{3\beta+11}{2\beta+1}, 4\right\} w_i + 2w_{i'}$. Let t_0, t_1, t_2 be the starting time of jobs j_0, j_1, j_2 by ADV, respectively.

- (a) **Case** $w_{j_1} > w_{i_0}$. Then by condition [B3] ($t = S_{i_0}$ and $\ell = j_1$), we have either $d_{j_1} \geq S_{i_0} + k + q_{j_1}$ or $d_{j_1} < S_{i_1} + k + q_{j_1}$. In the former, $w_{i'} \geq w_{j_1}$. Hence,

$$4w_{i_0} + 2w_{i'} > 4(2w_{i_1} + w_{i_2}) + w_{j_1} \geq 8w_{i_1} + 4w_{i_2} + w_{j_1}$$

However, as j_2, j_0 do not interrupt i_2, i_0 according to [B1] so $w_{j_2} \leq 2w_{i_2}$ and $w_{j_0} \leq 8w_{i_2}$. Therefore, $4w_{i_0} + 2w_{i'} > w_{j_0} + w_{j_1} + w_{j_2} + 2w_{i_2}$.

In the latter, job j_1 is urgent at $\max\{t_1, S_{i_1}\}$. Moreover, we know that no job ℓ satisfying $S_{i_1} + k + q_\ell \leq d_\ell < S_{i_0} + k + q_\ell$ and $w_\ell \geq w_{i_1}$ by definition of [B3]. So there is no job ℓ satisfying $S_{i_1} + k + q_\ell \leq d_\ell < \max\{t_1, S_{i_1}\} + k + q_\ell$ and $w_\ell \geq w_{i_1}$. As $w_{j_1} > w_{i_0}$, j_1 would have interrupted i_1 according to condition [B3] (contradiction).

- (b) **Case** $w_{j_1} \leq w_{i_0}$ and $w_{j_2} > w_{i_1}$. Then j_2 is not urgent at t_2 since otherwise, j_2 would have been scheduled instead of i_2 , contradicts to $w_{j_2} > w_{i_1}$. By condition [B2], $d_{j_2} \geq S_{i_1} + k + q_{j_2}$. Furthermore, by definition of condition [B3], we have $d_{j_2} \geq S_{i_0} + k + q_{j_2}$. Hence, $w_{i'} \geq w_{j_2} > w_{i_1} > w_{i_2}$. Therefore,

$$\begin{aligned} 4w_{i_0} + 2w_{i'} &> w_{j_1} + 3w_{i_0} + w_{j_2} + w_{i_2} \geq w_{j_1} + 3(2w_{i_1} + w_{i_2}) + w_{j_2} + w_{i_2} \\ &\geq w_{j_1} + 8w_{i_2} + w_{j_2} + 2w_{i_2} > w_{j_1} + w_{j_0} + w_{j_2} + 2w_{i_2} \end{aligned}$$

- (c) **Case** $w_{j_1} \leq w_{i_0}$ and $w_{j_2} \leq w_{i_1}$. We have:

$$\begin{aligned} w_{j_1} + w_{j_2} + w_{j_0} + 2w_{i_2} &\leq w_{i_0} + w_{i_1} + 8w_{i_2} + 2w_{i_2} \\ &\leq w_{i_0} + (w_{i_1} + \frac{1}{2}w_{i_2}) + \frac{19}{2(2\beta+1)}(2\beta+1)w_{i_2} \\ &\leq w_{i_0} + \frac{w_{i_0}}{2} + \frac{19}{2(2\beta+1)}w_{i_0} = \frac{3\beta+11}{2\beta+1} \cdot w_{i_0} \end{aligned}$$

where we use the facts that $w_{j_0} \leq 8w_{i_2}$ and $(2\beta+1)w_{i_2} \leq w_{i_0}$.

2. There are two interruptions of type [B3]

Let $i_0(= i), i_1, i_2, i_3$ be jobs started by ALG in the phase such that i_0, i_1, i_2 interrupt i_1, i_2, i_3 according to [B3], [B3] and [B2], respectively. Let j_0, j_1, j_2 and j_3 be marked jobs of intervals $[S_{i_0}, C_{i_0}), [S_{i_1}, S_{i_0}), [S_{i_2}, S_{i_1})$ and $[S_{i_3}, S_{i_2})$, respectively. By Lemma 1, the total charge from the beginning of the phase to S_{i_3} is at most $2w_{i_3} - 2w_t$. As job i is new released and urgent at time S_i , i receives no self-charge. In the following, we argue that $w_{j_3} + w_{j_2} + w_{j_1} + w_{j_0} + 2w_{i_3} \leq \max\left\{\frac{3\beta+11}{2\beta+1}, 4\right\} w_{i_0} + 2w_{i'}$. Let t_0, t_1, t_2, t_3 be the starting time of jobs j_0, j_1, j_2, j_3 by ADV, respectively.

The cases will be proceeded similarly as previous.

- (a) **Case** $w_{j_1} > w_{i_0}$. Then by condition [B3], we have either $d_{j_1} \geq S_{i_0} + k + q_{j_1}$ or $d_{j_1} < S_{i_1} + k + q_{j_1}$. In the former, $w_{i'} \geq w_{j_1}$. Hence,

$$4w_{i_0} + 2w_{i'} > 4(2w_{i_1} + w_{i_2}) + w_{j_1} \geq 8w_{i_1} + 4w_{i_2} + w_{j_1} \geq 20w_{i_2} + 8w_{i_3} + w_{j_1}$$

However, as j_3, j_2, j_0 do not interrupt i_3, i_2, i_0 according to [B1] so $w_{j_2} \leq 2w_{i_2}, w_{j_2} \leq 4w_{i_2}$ and $w_{j_0} \leq 16w_{i_2}$. Therefore, $4w_{i_0} + 2w_{i'} > w_{j_0} + w_{j_1} + w_{j_2} + w_{j_3} + 2w_{i_3}$, meaning that the total charge that i receives is at most $4w_i - 2w_t + 2w_{i'}$

In the latter, job j_1 is urgent at $\max\{t_1, S_{i_1}\}$. Moreover, we know that no job ℓ satisfying $S_{i_1} + k + q_\ell \leq d_\ell < S_{i_0} + k + q_\ell$ and $w_\ell \geq w_{i_1}$ by definition of [B3]. So there is no job ℓ satisfying $S_{i_1} + k + q_\ell \leq d_\ell < \max\{t_1, S_{i_1}\} + k + q_\ell$ and $w_\ell \geq w_{i_1}$. As $w_{j_1} > w_{i_0}$, j_1 would have interrupted i_1 according to condition [B3] (contradiction).

- (b) **Case** $w_{j_1} \leq w_{i_0}$ and $w_{j_2} > w_{i_1}$. Then j_2 is not urgent at t_2 since otherwise, j_2 would have been scheduled instead of i_2 , contradicts to $w_{j_2} > w_{i_1}$. By condition [B2], $d_{j_2} \geq S_{i_1} + k + q_{j_2}$. Furthermore, by definition of condition [B3], we have $d_{j_2} \geq S_{i_0} + k + q_{j_2}$. Hence, $w_{i'} \geq w_{j_2} > w_{i_1} > w_{i_2}$. Therefore,

$$\begin{aligned} 4w_{i_0} + 2w_{i'} &> w_{j_1} + 3w_{i_0} + w_{j_2} + w_{i_2} \geq w_{j_1} + 3(2w_{i_1} + w_{i_2}) + w_{j_2} + w_{i_2} \\ &\geq w_{j_1} + 8w_{i_2} + w_{j_2} + 2w_{i_2} > w_{j_1} + w_{j_0} + w_{j_2} + 2w_{i_2} \end{aligned}$$

- (c) **Case** $w_{j_1} \leq w_{i_0}$ and $w_{j_2} \leq w_{i_1}$. We have:

$$\begin{aligned} w_{j_1} + w_{j_2} + w_{j_0} + w_{j_3} + 2w_{i_3} &\leq w_{i_0} + w_{i_1} + 16w_{i_3} + 2w_{i_3} + 2w_{i_3} \\ &\leq w_{i_0} + w_{i_1} + \frac{40 - \beta}{5\beta + 2}w_{i_3} + \frac{101}{5\beta + 2}w_{i_2} \\ &\leq w_{i_0} + w_{i_1} + \frac{40 - \beta}{5\beta + 2}(2w_{i_2} + w_{i_3}) + \frac{21 + 2\beta}{5\beta + 2}w_{i_2} \\ &= w_{i_0} + \frac{4\beta + 42}{5\beta + 3}w_{i_1} + \frac{21 + 2\beta}{3\beta + 1}w_{i_2} \\ &\leq w_{i_0} + \frac{2\beta + 21}{5\beta + 3}(2w_{i_1} + w_{i_2}) \leq \frac{7\beta + 24}{5\beta + 3}w_{i_0} < \frac{3\beta + 11}{2\beta + 1}w_{i_0} \end{aligned}$$

where the third inequality is due to $w_{i_2} \geq \beta w_{i_3}$.

□