Improved Bounds for Facility Location Games with Fair Cost Allocation

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Abstract. We study Facility Location games played by n agents situated on the nodes of a graph. Each agent orders installation of a facility at a node of the graph and pays connection cost to the chosen node, and shares fairly facility installation cost with other agents having chosen the same location. This game has pure strategy Nash equilibria, that can be found by simple improvements performed by the agents iteratively. We show that this algorithm may need super-polynomial $\Omega(2^{n^{\frac{1}{2}}})$ steps to converge. For metric graphs we show that approximate pure equilibria can be found in polynomial time. On metric graphs we consider additionally *strong* equilibria; previous work had shown that they do not always exist. We upper bound the overall (social) cost of α-approximate strong equilibria within a factor $O(\alpha \ln \alpha)$ of the optimum, for every $\alpha \geq 1$.

1 Introduction

We study Facility Location games in which, self-interested agents situated on the nodes of a network request installation of facilities and connection to them. Each agent i may be associated with non-negative demand weight w_i and may order facility installation on any node v of the network. The agent tries to minimize his individual cost; it consists of the (weighted) distance of i from v and a *fair* share of the facility installation cost at v. Facility installation cost at v is shared evenly among all agents having chosen v, or proportionally to their demand weight in case of weighted agents. In this paper we study pure Nash and *strong* equilibria (PNE and SE) of the described facility location game, and improve or extend previous results [1, 2]. We prove a super-polynomial lower bound on the complexity of the *Iterative Best Response* algorithm for finding PNE, and analyze a polynomial-time algorithm for finding approximate PNE on metric networks. Also, for metric networks, we upper bound the *social cost* of approximate SE, that are resilient to coalitional deviations; the social cost is the sum of agents' individual costs. The upper bounding is with reference to the socially optimum cost; this ratio is known as the *Price of Anarchy* [3]. A previous result [2] provided an upper bound only in case of existence of exact SE.

The *fair* cost-sharing rule is of particular interest, as an instance of the *Shapley* Value that is stable for coalitional games [4]. The facility location game is a

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special case of the general network design game model with fair cost allocation proposed in [1], which has attracted a significant amount of recent research [5–8]. In this model each out of n selfish agents wishes to interconnect a given subset of network nodes, using network links. Each link is associated with a *cost* function and a *delay* function, both non-decreasing in the number of agents using the link. Agents try to minimize their individual cost, i.e. the sum of cost shares and delays of the links that an agent uses. For unweighted agents these games have PNE; they belong to the class of *potential* games, introduced by Monderer and Shapley [9]. They are associated with a *potential function* which, given any initial configuration of agents' strategies, can be optimized locally by a sequence of iterative improvements performed by the agents. When no agent can improve his cost any more, the resulting strategy configuration is a PNE. The authors in [1] studied the *Price of Stability* (*PoS*) of PNE, i.e. the worst-case ratio of the social cost of the least expensive equilibrium, relative to the socially optimum cost. They showed that for constant delay and cost functions $PoS = O(\ln n)$.

Recent work on network design with fair cost allocation has focused on identifying the Price of Stability, particularly when delays are zero. A lower bound of $\frac{4}{3}$ was given in [1], and improved to $\frac{12}{7}$ in [8]. The $O(\log n)$ upper bound for the general single-sink case was improved recently to $O(\frac{\log n}{\log \log n})$ [10]. Albers recently considered general network design with weighted players [5]; she proved an almost tight poly-logarithmic lower bound on the *PoS* for PNE and studied the Price of Anarchy of approximate SE with weighted and unweighted agents. SE of network design games were first studied by Epstein et al. [6]. The notion of SE is due to Aumann [11]. Metric Facility Location with unweighted players is the only case of the model of [1], in which tight constant bounds are known for the Price of Stability [2]. Also, existence of e-approximate SE ($e = 2.718\dots$) was shown in [2], even for the non-metric weighted case. Nguyen Kim [12] showed that PNE do not always exist for metric facility location games with weighted agents. Facility Location games have been used for modeling caching systems [13].

Contribution The complexity of finding PNE for network design games with fair cost allocation is an open problem. It was shown in [1] that $\Omega(2^{\frac{n}{3}})$ Iterative Best Responses may be needed before equilibrium is reached. This was shown though for the most general case of the game model. We prove that, even in the special case of Facility Location the algorithm needs super-polynomial $\Omega(2^{n^{\frac{1}{2}}})$ number of steps. On the positive side, we analyze a polynomial time algorithm for finding 2.258-*approximate* PNE on metric networks with uniform facility costs and weighted agents. This factor improves over the $e = 2.718\dots$ factor shown in [2] for general Facility Location games. Finally, we develop an analysis for the Price of Anarchy of α-approximate SE for metric Facility Location games with unweighted agents, and show that it is $SPoA = O(\alpha \ln \alpha)$. The $SPoA$ of exact strong equilibria was bound in [2] by a constant, only when they exist.

Definitions An instance of the facility location game is defined as a tuple (V, d, β, A, u, w). (V, d) defines a graph with vertex set V and distance function $d : V \times V \to \mathbb{R}_0^+ \cup \{\infty\}$. $\beta : V \to \mathbb{R}_0^+ \cup \{\infty\}$ associates every vertex with a facility opening cost. $A = \{1, \ldots, n\}$ is the set of agents residing on the graph, with $u : A \rightarrow V$ mapping agents to vertices, and $w : A \rightarrow \mathbb{R}^+$ associating agents with positive demand weight. We use β_v, u_i and w_i instead of $\beta(v)$, $u(i)$ and $w(i)$, respectively. Each agent is a player with strategy space V. Let $s_i \in V$ be the strategy of $i \in A$. A *strategy profile* is denoted by $s = (s_1, \ldots, s_n)$. We use s_{-i} for $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. $W_s(v) = \sum_{i:s_i=v} w_i$ is the sum of weights of agents playing v in s. The individual cost $c_i(s)$ of i under s is $c_i(s) = w_i d(u_i, s_i) + \frac{\beta_{s_i} w_i}{W_s(s_i)}$, i.e. i pays (weighted) connection cost and a fair share of the facility opening cost. Let $F_s = \{s_i | i \in A\}$. The *social cost* of s is defined as:

$$c(s) = \sum_{i \in A} c_i(s) = \sum_{i \in A} w_i d(u_i, s_i) + \sum_{v \in F_s} \beta_v$$

and coincides with the objective function of the facility location problem [14]. s is *socially optimum* if it is an optimum solution to the facility location problem.

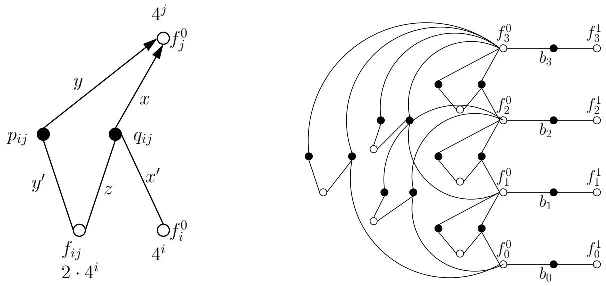
The game is metric when d is metric, unweighted when $w_i = 1$ for all $i \in A$, and has uniform facility costs when $\beta_v = 1$ for all $v \in V$. For $\alpha \geq 1$ a strategy profile s is an α-approximate pure Nash equilibrium (PNE) if for every player $i \in A$ and every strategy $s'_i \in V$ it is $c_i(s) \leq \alpha c_i(s_{-i}, s'_i)$. It is an α-approximate *strong* equilibrium (SE) if for every non-empty subset $I \subseteq A$, and pure strategies $s'_i \neq s_i$ for every $i \in I$, there is a $j \in I$ with $c_j(s) \leq \alpha c_j(s_{-I}, s'_I)$ [5, 6, 11]. When $\alpha = 1$, s is a PNE or a SE respectively.

For a strategy profile s, a *best response* of player i is any strategy s'_i that minimizes $c_i(s_{-i}, s'_i)$. The *Iterative Best Response* algorithm is initialized at an arbitrary profile s. It performs iteratively the following: while there is any player $i \in A$ and a best response s'_i of i with $c_i(s_{-i}, s'_i) > c_i(s)$, set s equal to (s_{-i}, s'_i). The algorithm terminates at a profile s which is a PNE [1, 9].

2 Complexity of Iterative Best Response

We derive a super-polynomial lower bound on the convergence complexity of the Iterative Best Response algorithm for non-metric facility location, later showing that it can be modified to work for metric facility location as well. We exhibit a family of instances of the game and a *schedule* of iterative best responses on this family, that simulates a *bit counter*. We assume n *bit* agents b_i, $i = 0 \ldots n-1$, each choosing among two facility nodes f_i^0 and f_i^1, referred to as the 0-strategy and 1-strategy of b_i. We use auxiliary agents, whose choices will enable the *bit* agents to switch between 0- and 1-strategies. For any other location in the graph we make the distance of b_i big enough, to discourage b_i from connecting there.

Every pair of bits i and $j > i$ will be grouped by a *gadget* G_{ij}, see fig. 1(a) for an illustration. G_{ij} has two auxiliary agent nodes q_{ij} (referred to as *inner* agent) and p_{ij} (referred to as outer agent) and one extra facility node f_{ij}. Agent q_{ij} is attached to facility nodes f_i^0, f_j^0, and f_{ij}. Agent p_{ij} is attached only to the facility nodes f_j^0 and f_{ij}. The total number of used gadgets is $O(n^2)$. Fig. 1(b) illustrates an instance with 4 bit agents. We introduce a schedule of Iterative



(a) A gadget G_{ij}.

(b) Construction with 4 bits.

Fig. 1: A gadget (initial state) and a construction of 4 bits, pairwise grouped by gadgets.

Best Response, formally described as algorithm 1, which we call `COUNT`. Initially every bit agent b_i is connected to f_i^0, and in every gadget G_{ij} agents p_{ij} and q_{ij} are connected to f_j^0. During execution of `COUNT` the changes occurring to p_{ij}, q_{ij} are depicted in order in fig. 2. Let us explain how `COUNT`(k) proceeds recursively. Initially, for every $j \in \{0, \ldots, k\}$, agents b_j, p_{ij} and q_{ij} for all $i < j$ are connected to f_j^0. The following high level steps occur:

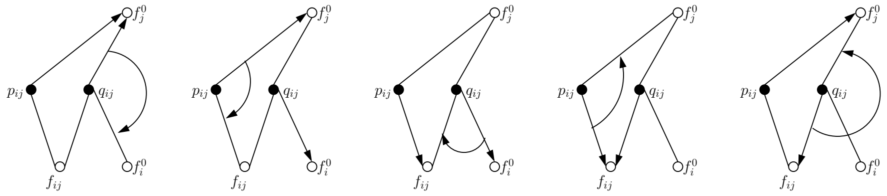
1. At first `COUNT`$(k-1)$ is called (line 1), that results in all agents b_i, $i < k$ being connected to f_i^1; also, agents p_{ji}, q_{ji} with $j < i$ are connected to f_{ji}.
2. Agent b_k switches from f_k^0 to f_k^1 (line 2). Lines 3-10 cause bit agents b_i, $i < k$ to switch back to f_i^0, so that number 2^{k-1} is formed by the strategy profile.
3. Finally, a call to `COUNT`$(k-1)$ is performed once more (line 11).

We explain lines 3-10 of the schedule. For any value of k, agent b_k is the first to leave f_k^0 and deviate to f_k^1. Agents q_{ik} follow, in order $i = k-1 \ldots 0$ and they deviate to f_i^0 (*Inner Fall* - fig. 2(a)). Each of them helps attracting sequentially *back to* f_i^0 all agents p_{ji}, q_{ji} in orders $j = i-1 \ldots 0$ and $j = 0 \ldots i-1$ respectively (*Outer* and *Inner Reset* - fig. 2(d) & (e)); recall that they were connected to f_{ji} after the high level step 1. Then, agents b_i, $i = k-1 \ldots 0$, are also attracted to f_i^0; at this time there is one agent more - q_{ij} - connected to f_i^0, than when b_i abandoned f_i^0 for f_i^1. Finally, this additional agent q_{ik} is attracted away from f_i^0, to create incentive for deviation to the bit agents b_i, $i < k$. This is achieved first by deviation of agents p_{ik} to f_{ik} in order $i = 0 \ldots k-1$ (*Outer Fall* - fig. 2(c)); this creates incentive for q_{ik} to leave f_i^0 for f_{ik} (*Inner Join* - fig. 2(d)).

In order for the moves specified by the schedule (algorithm 1) to be *best responses* of agents, we assign facility installation costs and distances so that certain inequalities hold. We assign f_i^0 cost 4^i, and facility nodes f_{ij} cost $2 \cdot 4^i$. Let distances be named as in fig. 1(a). The following inequalities should hold for the movements of a gadget's G_{ij} agents to be best responses. We consider movements of bit agents later. We use f for $2 \cdot 4^i$ - the cost of f_{ij}:

$$\textbf{Inner Fall} \text{ of } q_{ij}: \qquad x + \frac{4^j}{i+j+1} > x' + 4^i \quad , \; i = j-1 \ldots 0 \tag{1}$$

$$\textbf{Inner Fall} \text{ of } q_{ij}: \qquad z + f > x' + 4^i \quad , \; i = j-1 \ldots 0 \tag{2}$$



(a) **I**nner **F**all (b) **O**uter **F**all (c) **I**nner **J**oin (d) **O**uter **R**eset (e) **I**nner **R**eset

Fig. 2: Movements performed by the "inner" and "outer" agents of a gadget G_{ij}.

Algorithm 1 `COUNT`(k): counts up to $2^k - 1$.

```
if $k > 1$ then COUNT$(k-1)$                          {Count up to $2^{k-1}-1$}
$s_{b_k} \leftarrow f_k^1$                             {Set $b_k$ to 1}
for $i = k-1 \dots 0$ do
    $s_{q_{ik}} \leftarrow f_i^0$                      {Perform an Inner Fall for $q_{ik}$}
    for $j = i-1 \dots 0$ do $s_{p_{ji}} \leftarrow f_i^0$   {Perform an Outer Reset for $p_{ji}$}
    for $j = 0 \dots i-1$ do $s_{q_{ji}} \leftarrow f_i^0$   {Perform an Inner Reset for $q_{ji}$}
    $s_{b_i} \leftarrow f_i^0$                         {Set $b_i$ to 0}
end for
for $i = 0 \dots k-1$ do $s_{p_{ik}} \leftarrow f_{ik}$     {Perform an Outer Fall for $p_{ik}$}
for $i = 0 \dots k-1$ do $s_{q_{ik}} \leftarrow f_{ik}$     {Perform an Inner Join for $q_{ik}$}
if $k > 1$ then COUNT$(k-1)$                          {Count up to $2^{k-1}-1$}
```

$$\textbf{Outer Fall of } p_{ij}: \quad y + \frac{4^j}{j-i} > y' + f \quad , \ i = 0 \dots j-1 \tag{3}$$

$$\textbf{Inner Join of } q_{ij}: \quad x' + \frac{4^i}{2i+2} > z + \frac{f}{2} \quad , \ i = 0 \dots j-1 \tag{4}$$

$$\textbf{Outer Reset of } p_{ij}: \quad y' + \frac{f}{2} > y + \frac{4^j}{j-i+1} \ , \ i = j-1 \dots 0 \tag{5}$$

$$\textbf{Inner Reset of } q_{ij}: \quad z + f > x + \frac{4^j}{i+j+2} \ , \ i = 0 \dots j-1 \tag{6}$$

$$\textbf{Inner Reset of } q_{ij}: \quad x' + 4^i > x + \frac{4^j}{i+j+2} \ , \ i = 0 \dots j-1 \tag{7}$$

Correctness of inequalities (1)-(7) For (1) and (2) note that *Inner Falls* of q_{ij} agents start when b_j has already left f_j^0. Initially there are $2j+1$ agents connected to each f_j^0. Thus each q_{ij} for $i = j-1 \dots 0$ shares the cost of f_j^0 with $i+j+1$ agents exactly before deviating. Furthermore, it is the first to connect to f_i^0, and no agent is connected to f_{ij}. For (3) we note that the number of p_{ij} agents connected to f_j^0 is exactly $j-i$, for $i = 0 \dots j-1$. Agent b_j and all q_{ij} agents have already deviated before at the time when *Outer Falls* begin. Since the order by which they occur is $i = 0 \dots j-1$ (3) is justified; each p_{ij} is the first to connect to f_{ij}. For (4), note that q_{ij} has attracted back to f_i^0 the $2i+1$ agents connected to f_i^0 in the initial state. It is the second agent (after p_{ij}) to deviate to f_{ij}. The rest of the inequalities are justified if we consider that agents are reset in reverse order of their deviation, attracted to f_j^0 by an additional agent connected there, than when they deviated. To solve the system (1)-(7) at first we simplify as follows:

$$\begin{aligned} (3), (5) \quad &\Rightarrow \quad \frac{4^j}{j-i+1} - 4^i < y' - y < \frac{4^j}{j-i} - 2 \cdot 4^i && (8) \\ (1), (7) \quad &\Rightarrow \quad \frac{4^j}{i+j+2} - 4^i < x' - x < \frac{4^j}{i+j+1} - 4^i && (9) \\ (2), (4) \quad &\Rightarrow \quad \frac{(2i+1)4^i}{2i+2} < x' - z < 4^i && (10) \end{aligned}$$

The ranges given by (8)-(10) are non-empty. Several values for y,y' satisfy (8). For x',x and z we equalize $x' - x$, $x' - z$ to the average of bounds given by (9), (10) respectively. Then, we solve a system of 2 equations and 3 variables to express x' and z as functions of x. Sufficiently large $x > 0$, no larger than $O(4^n)$, ensures $x', z > 0$. This settles best responses of agents of G_{ij}. Let us decide distances of b_i from f_i^0 and f_i^1, and cost of f_i^1; b_i always leaves f_i^0 for f_i^1 when $2i$ other agents are connected to f_i^0, and returns to f_i^0 from f_i^1 when $2i+1$ other agents are connected to f_i^0. This yields 2 inequalities with 3 variables which can be easily satisfied. Facility nodes and distances other than the ones considered (fig. 1(b)) have a very big cost. The graph can be made metric by addition of a large constant to all described distances and usage of triangle inequality for the rest. Since we use $O(n^2)$ players we get:

Theorem 1. *Iterative Best Response for the Facility Location game with fair cost allocation may need superpolynomial* $\Omega(2^{n^{\frac{1}{2}}})$ *steps in the number of players.*

3 Approximation of Equilibria for Uniform Facility Costs

It is not known whether pure equilibria of facility location games can be computed efficiently. In light of Theorem 1 it seems reasonable to consider computation of approximate equilibria as a first step towards this goal. This was also suggested in [1] for the general network design model. In this section we restrict our attention to metric facility location games with uniform facility opening costs and describe an algorithm for computing approximate equilibria of such games. The existence of such an algorithm furthermore improves previous bounds for existence of approximate equilibria [2] in this setting when agents are weighted.

Let (V, d, β, A, u, w) be a metric weighted facility location game with uniform facility costs. Let $A(U) = \cup_{i:u_i \in U}\{i\}$ be the set of agents residing on nodes $v \in U$. We claim that Algorithm 2 constructs an approximate equilibrium s.

The parameter δ in the input of Algorithm 2 facilitates analysis of the algorithm. $S = \cup_{i \in A}\{u_i\}$ is the set of vertices where agents are residing. The algorithm incrementally constructs a set $F_s \subseteq S$ by adding more and more facilities. F_δ is maintained as the set of agents with a distance of at most $\frac{\delta}{w}$ to a facility $f \in F_s$, where w is the weight of the agent in question. While $S \setminus F_\delta \neq \emptyset$, the vertex of the agent in $S \setminus F_\delta$ with highest weight is added to F_s. Intuitively,

Algorithm 2 Compute an approximate equilibrium.

```
Input: $(V, d, \beta, A, u, w)$, $\delta$.
$F_s \leftarrow \emptyset$, $F_\delta \leftarrow \emptyset$, $S \leftarrow \cup_{i \in A}\{u_i\}$.
while $S \setminus F_\delta \neq \emptyset$ do
    Pick $i \in A(S \setminus F_\delta)$, such that $\forall j \in A(S \setminus F_\delta) : w_i \geq w_j$.
    $F_s \leftarrow F_s \cup \{u_i\}$.
    for $j \in A(S \setminus F_\delta)$ do
        if $w_j d(u_i, u_j) \leq \delta$ then $F_\delta \leftarrow F_\delta \cup \{u_j\}$.
    end for
end while
For all $i \in A : s_i \leftarrow \arg\min_{f \in F_s} d(u_i, f)$.
return $s = (s_1, \dots, s_n)$.
```

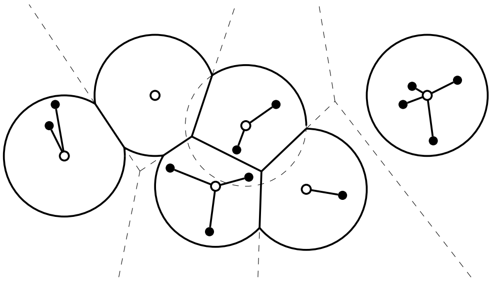


Fig. 3: Approximate equilibrium computed for an unweighted facility location game. The Voronoi diagram and the balls are included for illustrative purposes.

F_δ is the union of balls centered at the facilities of F_s. See Figure 3 for an illustration of the outcome for a facility location game with unweighted agents.

Theorem 2. *Algorithm 2 computes a* 2.258*-approximate equilibrium (*1.781*-approximate equilibrium) in* $O(n^2)$ *time given a metric weighted (unweighted) facility location game with uniform facility opening costs and* n *agents.*

Proof. First, let us observe that the running time of Algorithm 2 is at most quadratic in the number of agents. Indeed, at most $|S| \leq n$ facilities will be added to F_s, and finding the next facility and updating $S \setminus F_\delta$ can be done in $O(n)$ time. Connecting the agents to the nearest facility of F_s can be done in $O(n \cdot |F_s|)$ time, so overall the running time is $O(n^2)$.

Next, let us show that we have indeed computed a strategy profile s that is an approximate equilibrium. Let $i \in A$ be any agent. Let $f \in F_s$ be the facility that i is connected to in s, and let $f' \in F_s \cup \{u_i\}$ be a best response of i resulting in a strategy profile s'. Note that when i opens a facility on his own, i.e. $f' \in V \setminus F_s$, it is always a best response to open it at u_i. We need to show that $c_i(s)/c_i(s') \leq \alpha$ for some constant α. We can split this into the following two cases. We will use w_f, $f \in F_s$, to denote the weight of the agent that caused f to be picked for F_s.

Case 1: $f' = u_i \notin F_s$, i.e. agent i opens a facility on his own.

$$\frac{c_i(s)}{c_i(s')} = w_i d(u_i, f) + \frac{w_i}{W_s(f)} \leq w_i d(u_i, f) + \frac{w_i}{w_i + w_f} < \delta + \frac{2}{3} \tag{11}$$

In (11) we use $w_i d(u_i, f) \leq \delta$. Also, because addition of some facility g to F_s caused u_i to be covered by F_δ, we take 2 cases. Either $f = g$ and $w_i \leq w_f$, or $f \neq g$ and $w_i < 2w_f$ because $w_i d(u_i, g) \leq \delta < w_f d(f, g) \leq w_f(d(f, u_i) + d(u_i, g)) \leq w_f 2d(u_i, g)$. In both cases $w_i < 2w_f$, hence the last inequality.

Case 2. $f' \in F_s$, i.e. agent i connects to another facility that is already open.

$$\frac{c_i(s)}{c_i(s')} = \frac{w_i d(u_i, f) + w_i/W_s(f)}{w_i d(u_i, f') + w_i/W_{s'}(f')} \qquad < 1 + \frac{1}{w_f d(u_i, f')} \tag{12}$$

$$< 1 + 2/[w_f d(f, f')] \qquad < 1 + 2/\delta \tag{13}$$

In (12) we use that $d(u_i, f) \leq d(u_i, f')$ and that $w_f \leq W_s(f)$. In (13) we use the triangle inequality saying that $d(f, f') \leq d(f, u_i) + d(u_i, f') \leq 2d(u_i, f')$.

Hence, if $\alpha = \max\left(\delta + \frac{2}{3}, 1 + \frac{2}{\delta}\right)$, no agent will gain more than a factor of α by deviating. Then α is minimized for $\delta = \frac{1}{6}(1+\sqrt{73})$, with $\alpha = \frac{1}{6}(5+\sqrt{73}) < 2.258$.

If agents are unweighted, we would get $c_i(s)/c_i(s') \leq \delta + 1/2$ instead of (11) and $c_i(s)/c_i(s') < 1+1/\delta$ instead of (12). The second inequality follows from two subcases. Either i is the only agent going to f in s, in which case $c_i(s)/c_i(s') < 1 + 1/(w_f d(f, f'))$, or there is someone else sharing the cost, in which case $c_i(s)/c_i(s') < 1 + 1/(2w_f d(u_i, f'))$. Hence, we could let $\alpha = \max\left(\delta + \frac{1}{2}, 1 + \frac{1}{\delta}\right)$, which is minimized for $\delta = (1 + \sqrt{17})/4$, with $\alpha = (3 + \sqrt{17})/4 < 1.781$. □

Approximate strong equilibria are also approximate pure equilibria, by definition. The following corollary improves over the best known approximation factor of $e = 2.718\ldots$, for approximate pure equilibria with weighted agents [2].

Corollary 1. 2.258-*approximate equilibria are guaranteed to exist in weighted metric facility location games with uniform facility opening costs.*

4 Strong Price of Anarchy: The Unweighted Metric Case

The metric unweighted Facility Location game does not always have exact SE, but α-approximate SE always exist, for $\alpha \geq e = 2.718\ldots$ [2]. We prove an upper bound for the Price of Anarchy of α-approximate SE (Strong Price of Anarchy - $SPoA_\alpha$). Exact SE were shown to have $SPoA = O(1)$, when they exist [2]. Let s be any α-approximate SE, s^* the socially optimum profile, and $A_{s^*}(v)$ the subset of agents that are connected to $v \in F_{s^*}$ under s^*. For $v \in F_{s^*}$ define $c_v(s^*) = \beta_v + \sum_{i:s_i^*=v} d(u_i, v)$. We upper bound the $SPoA_\alpha$ as:

$$SPoA_\alpha = \frac{c(s)}{c(s^*)} = \frac{\sum_{v \in F_{s^*}} \sum_{i \in A_{s^*}(v)} c_i(s)}{\sum_{v \in F_{s^*}} c_v(s^*)} \leq \max_{v \in F_{s^*}} \frac{\sum_{i \in A_{s^*}(v)} c_i(s)}{c_v(s^*)}$$

We will upper bound the latter ratio for any $v \in F_{s^*}$, so as to prove:

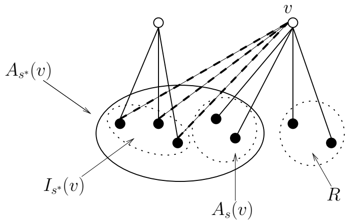


Fig. 4: The situation examined in the proof of theorem 3.

Theorem 3. *For any $\alpha \geq 1$, the Price of Anarchy of α-approximate strong equilibria in the unweighted metric Facility Location game is $O(\alpha \ln \alpha)$.*

For any facility node $v \in F_{s^*}$, define $A_s(v) \subseteq A_{s^*}(v)$ to be the subset of those agents that are connected to v both in s^* and s. Define $I_{s^*}(v) = A_{s^*}(v) \setminus A_s(v)$ to be the subset of agents that are connected to v in s^*, but not in s. Let R denote agents connected to v under s, with $R \subseteq A \setminus A_{s^*}(v)$. See fig. 4 for an illustration. To simplify notation we use $d(u_i, v) = x_i^*$, for $i \in A_{s^*}(v)$. With respect to R we will consider two cases. At first assume that, under s_{-R}, no agent of $I_{s^*}(v)$ has incentive to deviate to v in with coalition $I_{s^*}(v)$. Then for every $i \in I_{s^*}(v)$ it is $c_i(s) \leq \alpha c_i(s^*)$, because s is an α-approximate SE. Then:

$$\sum_{i \in A_{s^*}(v)} c_i(s) = \sum_{i \in A_s(v)} c_i(s) + \sum_{i \in I_{s^*}(v)} c_i(s) \leq \beta_v + \sum_{i \in A_s(v)} x_i^* + \alpha \left(\beta_v + \sum_{i \in I_{s^*}(v)} x_i^* \right)$$

Thus $\sum_{i \in A_{s^*}(v)} c_i(s) \leq (1+\alpha) \left(\beta_v + \sum_{i \in A_{s^*}(v)} x_i^* \right) = (1+\alpha) c_v(s^*)$.

For the rest of the analysis we assume there exists at least one agent $i \in I_{s^*}(v)$ willing to deviate to v in coordination with the coalition $I_{s^*}(v)$ under s_{-R}. Define a *minimal disagreeing subset* $I^0_{s^*}(v) \subseteq I_{s^*}(v)$ as a *minimal subset of misconnected agents* containing an agent i that would deviate to v with $I^0_{s^*}(v)$, under s_{-R}. Also define $J_{s^*}(v) = I_{s^*}(v) \setminus I^0_{s^*}(v)$. Fix $i \in I^0_{s^*}(v)$ to be from now on the agent (or one of them if there are many) that would deviate to v in coordination with $I^0_{s^*}(v)$. We call i the *unstable* agent of the minimal disagreeing subset $I^0_{s^*}(v)$. By definition, the following holds for the *unstable* agent i:

$$c_i(s) > \alpha \left(x_i^* + \frac{\beta_v}{|I^0_{s^*}(v)| + |A_s(v)|} \right) \tag{14}$$

The rest of the analysis consists of bounds for agents in $I^0_{s^*}$ and $J_{s^*}(v)$ separately. In particular, lemmas 1 and 2 describe upper bounds for these sets respectively.

Lemma 1. *Let $I_{s^*}(v)$ be a subset of misconnected agents under an α-approximate strong equilibrium profile s. Let $I^0_{s^*}(v)$ be a minimal disagreeing subset of $I_{s^*}(v)$ and $i \in I^0_{s^*}(v)$ an unstable agent. Then:*

$$\sum_{l \in I^0_{s^*}(v)} c_l(s) \leq 2\alpha \Big(\beta_v + \sum_{l \in I^0_{s^*}(v)} x_l^* \Big) \tag{15}$$

Proof. By minimality of $I^0_{s^*}(v)$, every agent $l \in I^0_{s^*}(v)$ is *not willing* to deviate to v under s_{-R} with a coalition of size $|I^0_{s^*}(v)| - 1$. Thus for every $l \in I^0_{s^*}(v)$:

$$c_l(s) \leq \alpha \left(x^*_l + \frac{\beta_v}{|I^0_{s^*}(v)| + |A_s(v)| - 1} \right) \leq \alpha \left(x^*_l + \frac{\beta_v}{|I^0_{s^*}(v)| - 1} \right)$$

Summing over $I^0_{s^*}(v)$ yields the upper bound of (15). □

Lemma 2. *Let $I_{s^*}(v)$ be a subset of misconnected agents under an α-approximate strong equilibrium profile s. Let $I^0_{s^*}(v)$ be a minimal disagreeing subset of $I_{s^*}(v)$ and $J_{s^*}(v) = I_{s^*}(v) \setminus I^0_{s^*}(v)$. Then:*

$$\sum_{j \in J_{s^*}(v)} c_j(s) \leq \alpha \sum_{j \in J_{s^*}(v)} x^*_j + \alpha \beta_v \Big(H(|A_{s^*}(v)|) - H(|I^0_{s^*}(v)| + |A_s(v)|) \Big) \quad (16)$$

Proof. We use an argument by Albers [5]. W.l.o.g. name agents $j \in J_{s^*}(v)$ by distinct indices $1, \ldots, |J_{s^*}(v)|$ and define a series of supersets of $I^0_{s^*}(v)$, as follows: $I^j_{s^*}(v) = I^{j-1}_{s^*}(v) \cup \{j\}$. Since s is an α-approximate SE, every set $I^j_{s^*}(v)$ contains an agent not willing to deviate to v in coordination with $I^j_{s^*}(v)$; it is found either in $I^0_{s^*}(v) \setminus \{i\}$ or in $I^j_{s^*}(v) \setminus I^0_{s^*}(v)$. We can assume w.l.o.g. that for $I^j_{s^*}(v)$ this agent is j; otherwise we exchange j with an agent from $I^0_{s^*}(v) \setminus \{i\}$. By definition of a minimal disagreeing subset, this will not harm our previous results. Then:

$$c_j(s) \leq \alpha \Big(x^*_j + \frac{\beta_v}{|I^j_{s^*}(v)| + |A_s(v)| + |R|} \Big), \ \ j = 1, \ldots, |J_{s^*}(v)| \in J_{s^*}(v) \quad (17)$$

We omit $|R|$ and sum the inequality over $j \in J_{s^*}(v)$. The result follows. □

The following lemma will provide a lower bound for the socially optimum connection cost $\sum_{j \in J_{s^*}(v)} x^*_j$ appearing in the upper bounding expression (16).

Lemma 3. *Let $I_{s^*}(v)$ be a subset of misconnected agents under α-approximate strong equilibrium profile s. Let $I^0_{s^*}(v)$ be a minimal disagreeing subset of $I_{s^*}(v)$ and $J_{s^*}(v) = I_{s^*}(v) \setminus I^0_{s^*}(v)$. Then for $r = |I^0_{s^*}(v)| + |A_s(v)|$:*

$$\sum_{j \in J_{s^*}(v)} x^*_j \geq \frac{\beta_v}{1 + \alpha} \Big(\frac{|A_{s^*}(v)| - \lceil \alpha r \rceil}{r} - \alpha \Big(H(|A_{s^*}(v)|) - H(\lceil \alpha r \rceil) \Big) \Big) \quad (18)$$

Proof. Let i be the fixed *unstable* agent of $I^0_{s^*}(v)$. Note that, under strategy profile s, i does not have an incentive to join facility node s_j for any $j \in J_{s^*}(v)$. Thus if j pays for s_j a share of $\frac{\beta_{s_j}}{\lambda_j}$ (that is, s_j serves λ_j agents in total in s):

$$c_i(s) \le \alpha\left(d(u_i,s_j) + \frac{\beta_{s_j}}{1+\lambda_j}\right) \le \alpha\left(d(u_i,v) + d(u_j,v) + d(u_j,s_j) + \frac{\beta_{s_j}}{\lambda_j}\right) \Rightarrow$$

$$c_i(s) \le \alpha\left(x_i^* + x_j^* + c_j(s)\right) \le \alpha\left(x_i^* + x_j^* + \alpha\left(x_j^* + \frac{\beta_v}{|I^j_{s^*}(v)| + |A_s(v)|}\right)\right) \tag{19}$$

The last inequality derives by (17) for $c_j(s)$ and by safely omitting $|R|$. Using (19) and the lower bound for $c_i(s)$ by (14), we solve for x_j^*. By definition of $I^j_{s^*}(v)$ in lemma 2, $|I^j_{s^*}(v)| = |I^0_{s^*}(v)| + j$, $j = 1, \ldots, |J_{s^*}(v)|$. Then for $j = 1 \ldots |J_{s^*}(v)|$:

$$x_j^* \ge \max\Big\{0, \frac{\beta_v}{1+\alpha}\Big(\frac{1}{|I^0_{s^*}(v)| + |A_s(v)|} - \frac{\alpha}{j + |I^0_{s^*}(v)| + |A_s(v)|}\Big)\Big\}$$

Finally we sum up the latter bound over all j. Notice that x_j^* becomes non-zero only when $j + |I^0_{s^*}(v)| + |A_s(v)| \ge \alpha(|I^0_{s^*}(v)| + |A_s(v)|)$. Since $j + |I^0_{s^*}(v)| + |A_s(v)|$ is an integral value, it turns out that x_j^* becomes non-negative for those values of j for which it is $j + |I^0_{s^*}(v)| + |A_s(v)| \ge \lceil\alpha(|I^0_{s^*}(v)| + |A_s(v)|)\rceil$. Then, by setting $r = |I^0_{s^*}(v)| + |A_s(v)|$, and by summing up over all j we obtain (18). ☐

Proof of Theorem 3 We put everything together. A *lower bound* on $c_v(s^*)$ is:

$$c_v(s^*) \ge \beta_v + \sum_{j \in J_{s^*}(v)} x_j^* + \sum_{l \in I^0_{s^*}(v)} x_l^* + \sum_{i \in A_s(v)} x_i^* \tag{20}$$

Accordingly, we obtain the following *upper bound* on $\sum_{i \in A_{s^*}(v)} c_i(s)$ by (15):

$$\begin{aligned}\sum_{i \in A_{s^*}(v)} c_i(s) &\le \sum_{l \in I^0_{s^*}(v)} c_l(s) + \sum_{j \in J_{s^*}(v)} c_j(s) + \sum_{i \in A_s(v)} c_i(s) \\ &\le 2\alpha\Big(\beta_v + \sum_{l \in I^0_{s^*}(v)} x_l^*\Big) + \sum_{j \in J_{s^*}(v)} c_j(s) + \sum_{i \in A_s(v)} c_i(s) \qquad (21)\end{aligned}$$

We use (20), (21) for bounding $\frac{1}{c_v(s^*)} \sum_{i \in A_{s^*}(v)} c_i(s)$, and substitute by (16), (18). Using bounds $\gamma + \ln m \le H(m) \le 1 + \ln m$ for the harmonic numbers ($\gamma > 0.5$ is Euler' constant), and $\lceil \alpha r \rceil \le (1+\alpha)r$, we obtain:

$$SPoA \le 1 + 2\alpha + 2\alpha \frac{1 - \gamma + \ln \frac{|A_{s^*}(v)|}{r}}{\frac{1}{1+\alpha}\Big(\frac{|A_{s^*}(v)|}{r} - \alpha \ln \frac{|A_{s^*}(v)|}{r} + \alpha(\gamma + \ln\alpha - 1)\Big)}$$

We substitute γ with 0.5 and, by setting $y = \frac{|A_{s^*}(v)|}{r}$, we simplify to:

$$SPoA_\alpha < 1 + 2\alpha + 2\sqrt{e}(1+\alpha)\frac{\ln\alpha + \ln(y\sqrt{e}/\alpha)}{y\sqrt{e}/\alpha - \sqrt{e}\ln(y\sqrt{e}/\alpha)}$$

For $x > 0$ it is $\frac{1}{x - \sqrt{e}\ln x} = O(1)$ and $\frac{lnx}{x - \sqrt{e}\ln x} = O(1)$, thus $SPoA_\alpha = O(\alpha \ln \alpha)$.

Corollary 2. *The Price of Anarchy of* e*-approximate strong equilibria for the metric unweighted Facility Location game is* $O(1)$.

5 Future work

Computing PNE and approximate strong equilibria remains wide open. Hardness results in this respect would be of major interest as they would not only apply to facility location, but also to more general models. Besides being of independent interest, further improvements of approximation algorithms might lead to insights in this area. In particular, one might produce a better way of assigning agents to facilities in Algorithm 2. Studying the quality of computed (approximate) equilibria would be of importance as well. A wide range of approximation algorithms are known for the classical facility location problem, some of which might be modified to produce approximate equilibria of low cost. Finally, the existence of PNE of weighted facility location games with uniform facility costs remains unresolved.

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