Formal topology applied to Riesz spaces

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mostly jww Thierry Coquand
Problem 1
Gian-Carlo Rota (similar remarks by Kolmogorov)
(‘Twelve problems in probability no one likes to bring up’)
Number 1: ‘The algebra of probability’
About the pointwise definition of probability:
‘The beginning definitions in any field of mathematics are always misleading, and the basic definitions of probability are perhaps the most misleading of all.’
Problem: Probability should not be build up from points: impossible events! → develop ‘pointless probability’ (work by Caratheory and von Neumann)
von Neumann - towards Quantum Probability
Constructive maths

Constructive mathematics
Two important interpretations:

1. Computational: type theory, realizability, Eff, ...
2. Geometrical: (sheaf) toposes, ...

Research in constructive maths (analysis) mainly focuses on 1, where we have DC
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Problem 2
Develop constructive maths without (countable) choice

Richman

‘Measure theory and the spectral theorem are major challenges for a choiceless development of constructive mathematics and I expect a choiceless development of this theory to be accompanied by some surprising insights and a gain of clarity.’

We will address both of these problems simultaneously.
Choice is used to construct *ideal* points (real numbers, max. ideals). Avoiding points one can avoid choice and non-constructive reasoning

- Pointfree topology aka locale theory, formal topology (formal opens)

These formal objects model basic observations

Topology: lattice of sets closed under unions and finite intersections
Pointfree topology: lattice closed under joins and finite meets
pointfree topology = complete Heyting algebra
Constructive integration theory

See Palmgren’s talk.

- Riemann
- Lebesgue
- Daniell - Positive linear functionals
  - Bishop integration spaces
Riemann considered partitions of the domain

\[ \int f = \lim \sum f(x_i)|x_{i+1} - x_i| \]
Lebesgue considered partitions of the range

\[ \int f = \lim \sum s_i \mu(s_i \leq f < s_{i+1}) \]
Consider integrals on algebras of functions.

Classical Daniell theory
integration for positive linear functionals on space of continuous functions on a topological space
Prime example: Lebesgue integral $\int$
Linear: $\int af + bg = a \int f + b \int g$
Positive: If $f(x) \geq 0$ for all $x$, then $\int f \geq 0$. 
Consider integrals on algebras of *functions*.

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Other example: Dirac measure \( \delta_t(f) := f(t) \).
Consider integrals on algebras of functions. Classical Daniell theory integration for positive linear functionals on space of continuous functions on a topological space.
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Other example: Dirac measure \( \delta_t(f) := f(t) \).
Can be extended to a quite general class of underlying topological spaces.
Bishop’s integration theory

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\[ C(X) \rightarrow L_1 \]
\[ \downarrow \quad \downarrow \]
\[ L_1 \]

$L_1$: concrete functions.

$L_1$: $\mathcal{L}_1$ module equal almost everywhere.
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\[
\begin{array}{ccc}
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\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
L_1 & & L_1
\end{array}
\]

\( \mathcal{L}_1 \): concrete functions

\( L_1 \): \( \mathcal{L}_1 \) module equal almost everywhere

Work with \( \mathcal{L}_1 \) because functions ‘are easy’.

Secretly we work with \( L_1 \).

Do this overtly with an abstract space of functions, see later.
We generalize Bishop/Cheng and metric Boolean algebras

**Definition**

A *Riesz space* (vector lattice) is a vector space with ‘compatible’ lattice operations ∨, ∧.

E.g. \( f \vee g + f \wedge g = f + g \).

Prime (‘only’) example:
vector space of real functions with pointwise ∨, ∧.
Also: the simple functions.
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A Riesz space (vector lattice) is a vector space with ‘compatible’ lattice operations $\vee$, $\wedge$.
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vector space of real functions with pointwise $\vee$, $\wedge$.
Also: the simple functions.
We assume that Riesz space $R$ has a strong unit 1: $\forall f \exists n. f \leq n \cdot 1$.
An integral on a Riesz space is a positive linear functional $I$.
Most of Bishop’s results generalize to Riesz spaces!
However, we first need to show how to handle multiplication.
Once we know how to do this we can treat:

1. integrable, measurable functions, $L_p$-spaces
2. Riemann-Stieltjes
3. Dominated convergence
4. Radon-Nikodym
5. Spectral theorem
Profile theorem

The profile theorem is crucial in Bishop’s development. However, it implies that the reals are uncountable.

**Theorem (Rosolini/S)**

*The (Dedekind) reals are not uncountable (in \( Sh(\mathbb{R}) \)).*

i.e. can not be proved with CAC
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‘Every Riesz space can be embedded in an algebra of continuous functions’

Theorem (Classical Stone-Yosida)

Let $R$ be a Riesz space. Let $\operatorname{Max}(R)$ be the space of representations. The space $\operatorname{Max}(R)$ is compact Hausdorff and there is a Riesz embedding $\hat{\cdot} : R \rightarrow C(\operatorname{Max}(R))$. The uniform norm of $\hat{a}$ equals the norm of $a$.

We will replace $\operatorname{Max}(R)$ by a formal space.

- Substitute for the profile theorem
- Towards spectral theorem
- To define multiplication
Pointfree definition of a space using entailment relation $\vdash$
Used to represent distributive lattices
Write $A \vdash B$ iff $\land A \leq \lor B$
Conversely, given an entailment relation define a lattice:
Lindenbaum algebra
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Topology is a distributive lattice

order: covering relation

‘Domain theory in logical form’

Topology = theory of (finite) observations (Smyth, Vickers, Abramsky ...)
Pointfree definition of a space using entailment relation $\vdash$
Used to represent distributive lattices
Write $A \vdash B$ iff $\wedge A \leq \bigvee B$
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Lindenbaum algebra
Topology is a distributive lattice
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`Domain theory in logical form'
Topology = theory of (finite) observations (Smyth, Vickers, Abramsky ...)
Stone’s duality:
Boolean algebras and Stone spaces
distributive lattices and coherent $T_0$ spaces
Points are models
space is theory, open is formula
model theory $\rightarrow$ proof theory
... and the classical theorem (by a direct application of AC). Bishop proves the representation theorem using $\epsilon$-eigenvalues, which has computational content, to prove that a bound is preserved, which has no computational content. We avoid such excursions.

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Bishop proves the representation theorem using $\epsilon$-eigenvalues, which has computational content, to prove that a bound is preserved, which has no computational content.
We avoid such excursions.
We have proved the Stone-Yosida representation theorem:

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**Theorem**

*Every Riesz space can be embedded in an algebra of continuous functions on its spectrum qua formal space.*

Any integral can be extended to all the continuous functions. Thus we are in a formal Daniell setting!
We can now develop much of Bishop’s integration theory in this abstract setting.
An f-algebra is a Riesz space with multiplication.

**Theorem**

*Every f-algebra is commutative.*

Several proofs using AC.
‘Constructive’ (i.e. no AC) proof by Buskens and van Rooij.
Mechanically translation to a *simpler constructive* proof (no PEM, AC) which is entirely internal to the theory of Riesz spaces.
Summary

- Observational mathematics
  - Topology
  - Measure theory
- Integration on Riesz spaces (towards Richman’s challenge).
  - ‘functions’ instead of ‘opens’
  - Most of Bishop’s results can be generalized to this setting!
- New (easier) proof of Bishop’s spectral theorems using Coquand’s Stone representation theorem (pointfree topology)
- The reals are not uncountable.
- Pointfree is natural in constructive maths without choice
- There’s more...
- Formal Topology and Constructive Mathematics: the Gelfand and Stone-Yosida Representation Theorems (with Coquand)
- Constructive algebraic integration theory without choice
- Constructive algebraic Integration theory
- Coquand - About Stone’s notion of spectrum J. Pure Appl. Algebra, 197(1-3):141-158, 2005
Constructive mathematics (Brouwer, Markov, Bishop, ...) mostly deals with complete separable metric spaces, images of Baire space ($\mathbb{N}^\mathbb{N}$ with product topology)

Example: $[0, 1]$ limits of Cauchy sequences/ image of $3^\mathbb{N}$

see also reverse maths, explicit maths, Weihrauch’s TTE

Has surprisingly large range, but invites sequential reasoning (representation dependent)
Richman: DC is often used to pick a path (choice sequence) in a tree/subset of Baire space.
Proposal: consider the trees of all paths directly.
Example: construction of all zeros of a polynomial in the FTA.
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Proposal: consider the trees of all paths directly.
Example: construction of all zeros of a polynomial in the FTA.
The tree represents a topological space.
Here we give a formal description of this space.
Basic opens for finite paths.
Now: consider the formal space of ‘all’ choices.
Again the idea was obtained in both worlds:
Brouwer’s theory of spreads and in topos theory