Comparator Networks for Binary Heap Construction

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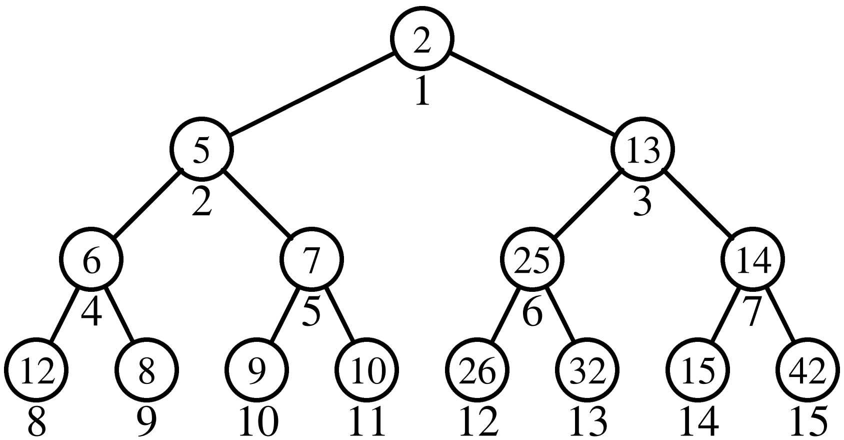
Binary Heap

– Williams 1964



2	5	13	6	7	25	14	12	8	9	10	26	32	15	42
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

Priority queue

– Insert, DeleteMin $\mathcal{O}(\log n)$

Sorting $\mathcal{O}(n \log n)$

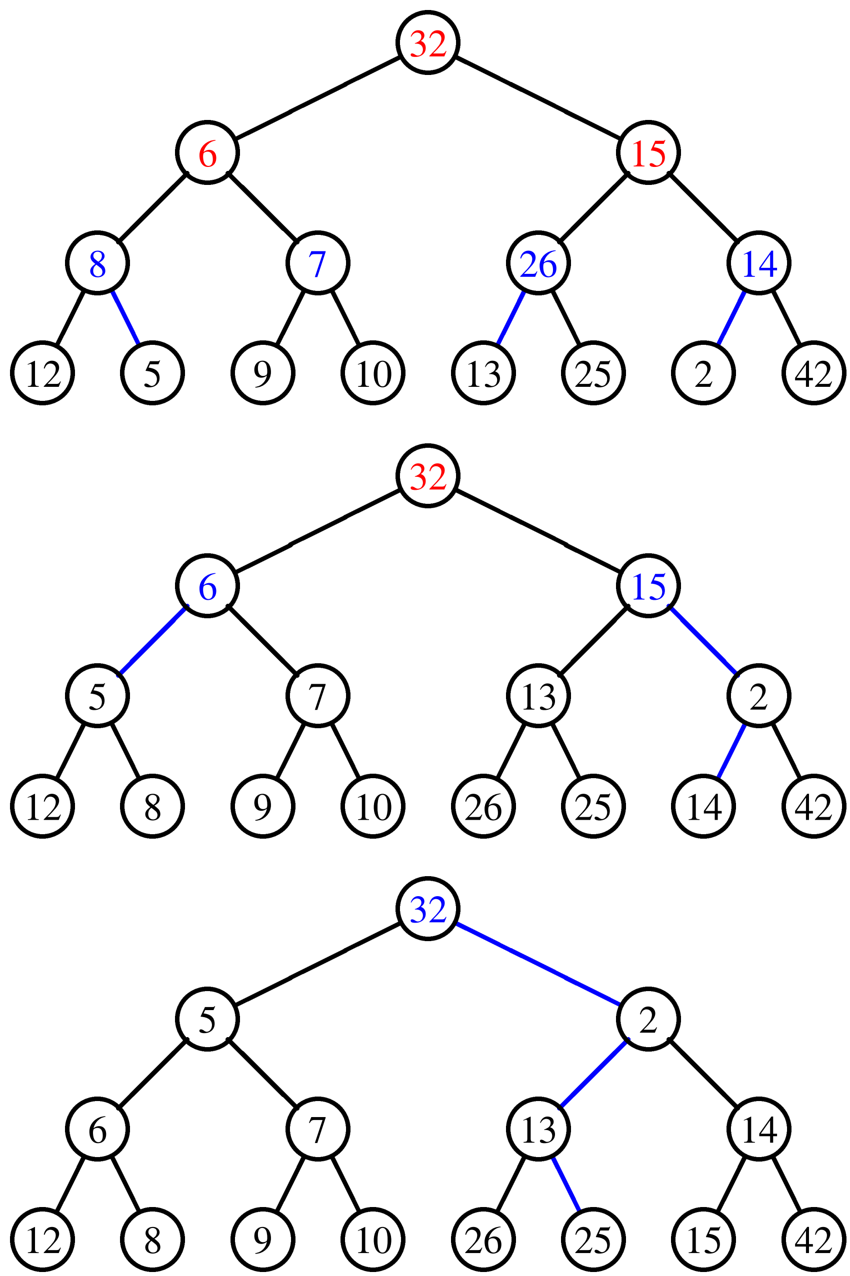
Construction $\mathcal{O}(n)$

Selecting the t smallest $\mathcal{O}(t)$

Implicit representation

The Sequential Binary Heap Construction Algorithm

– Floyd 1964



Alg. For each node $i = n, \ldots, 1$ apply $\mathsf{SiftDown}(i)$

Binary Heap Construction

Sequential

– $\mathcal{O}(n)$ *– Floyd 1964*

Optimal parallel algorithms

– EREW PRAM: $\mathcal{O}(\log n)$ *– Olariu, Wen 1991*

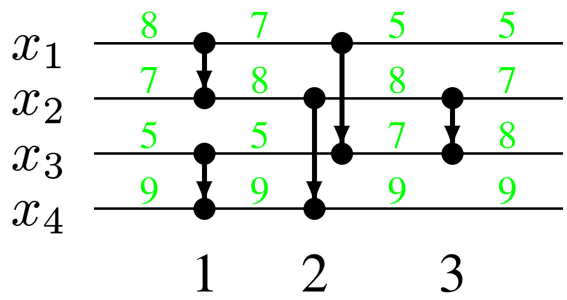
– CRCW PRAM: $\mathcal{O}(\log \log n)$ *– Dietz, Raman 1994*

– Randomized parallel comparison tree model: $\mathcal{O}(\alpha(n))$ *– Dietz 1992*

Question

– Comparator networks (the simplest parallel model of computation) ?

Comparator Networks



An optimal sorting network for $n = 4$

Merging networks: $\mathcal{O}(n \log n)$ *– Batcher 1960's*

Sorting networks: $\mathcal{O}(n \log^2 n)$ *– Batcher 1960's*

AKS sorting networks: $\mathcal{O}(n \log n)$ *– Ajtai, Komlós, Szemerédi 1983*

Median selection networks: $\mathcal{O}(n \log n)$ *– Ajtai, Komlós, Szemerédi 1983*

Sorting, merging, and median require networks of size $\Omega(n \log n)$
– Alekseev 1969
– Miltersen, Paterson, 1996

Note Sequentially merging and median can be solved in linear time
– Blum, Floyd, Pratt, Rivest, Tarjan 1973

Binary Heaps and Comparator Networks

Heap Construction Networks

– Comparator networks that permute n inputs to an implicit heap

– $H(n)$ = minimum size of a heap construction network

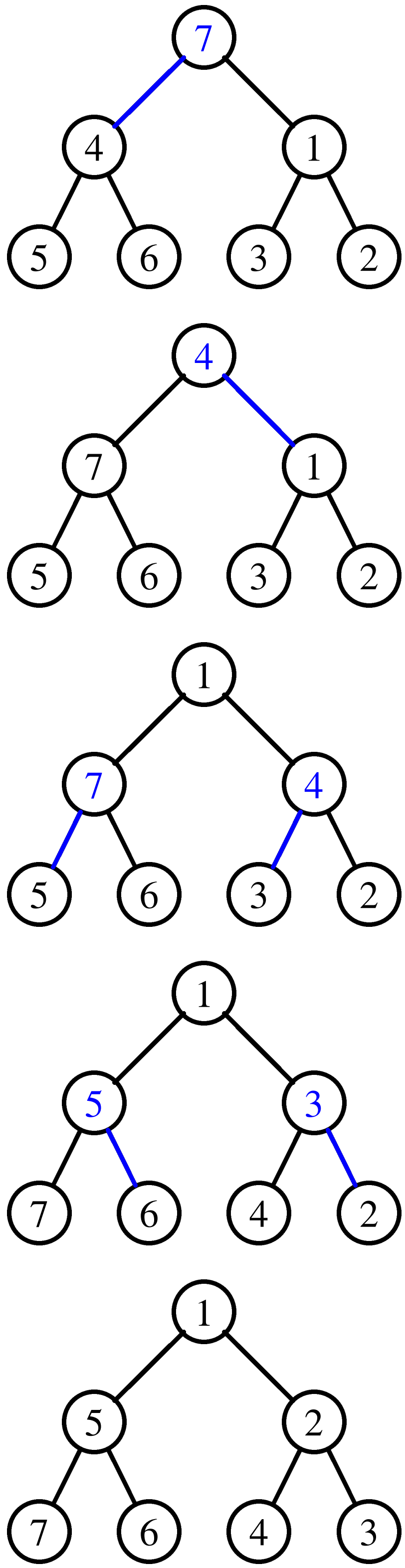
– $H(n) = \mathcal{O}(n \log n)$ because of $\mathcal{O}(n \log n)$ sorting networks

(n, t)-Selection Networks

– Comparator networks that select the t smallest elements from n inputs

– $S(n, t)$ = minimum size (n, t)-selection network

$\mathcal{O}(n \log n)$ Heap Construction Networks



SiftDown networks

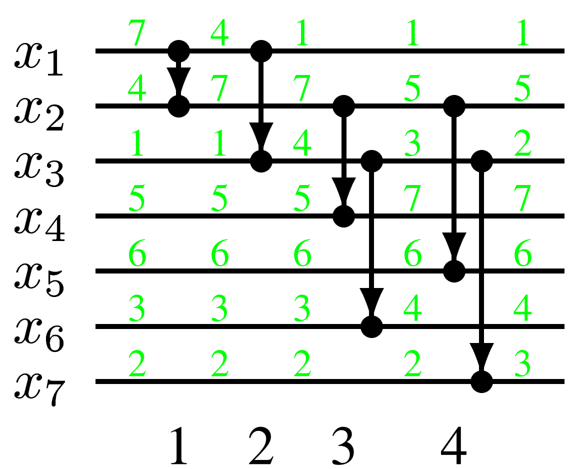
- Each tree edge ≡ comparator
- Size $n-1$, depth $\mathcal{O}(\log n)$

Heap construction network

- Size

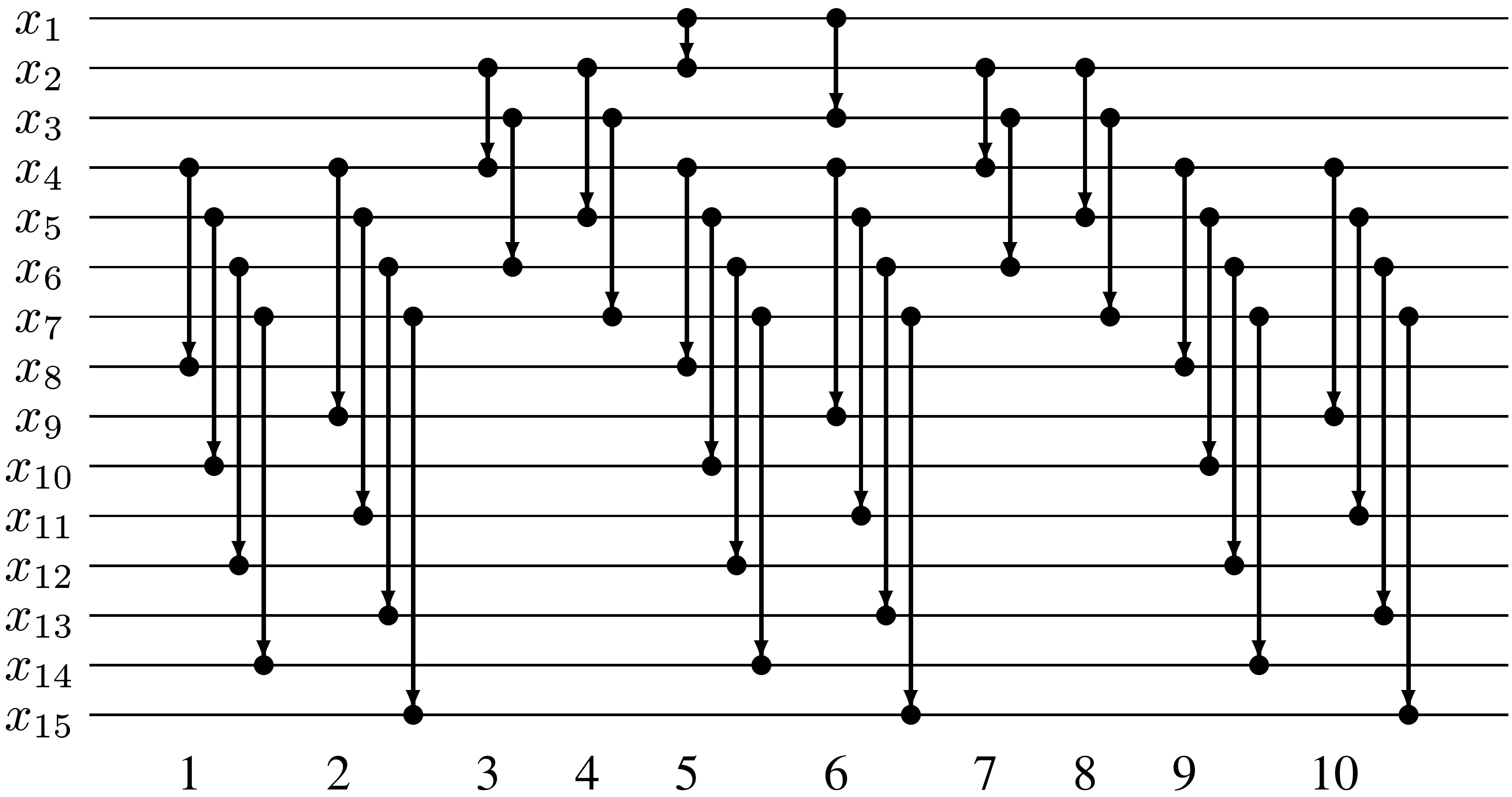
$$\mathcal{O}\left(\sum_{i=0}^{\log n} 2^i \cdot \frac{n}{2^i}\right) = \mathcal{O}(n \log n)$$

- Depth $\mathcal{O}(\log n)$



SiftDown network, $n = 7$

$\mathcal{O}(n \log n)$ **Heap Construction Network**



$n = 15$ Size $= 34$ Depth $= 10$

$\mathcal{O}(n \log\log n)$ Heap Construction Networks

Lemma $S(n, \lfloor \frac{n}{2} \rfloor) \leq Cn \log n + \mathcal{O}(n)$, $C > \frac{3}{\log 3}$

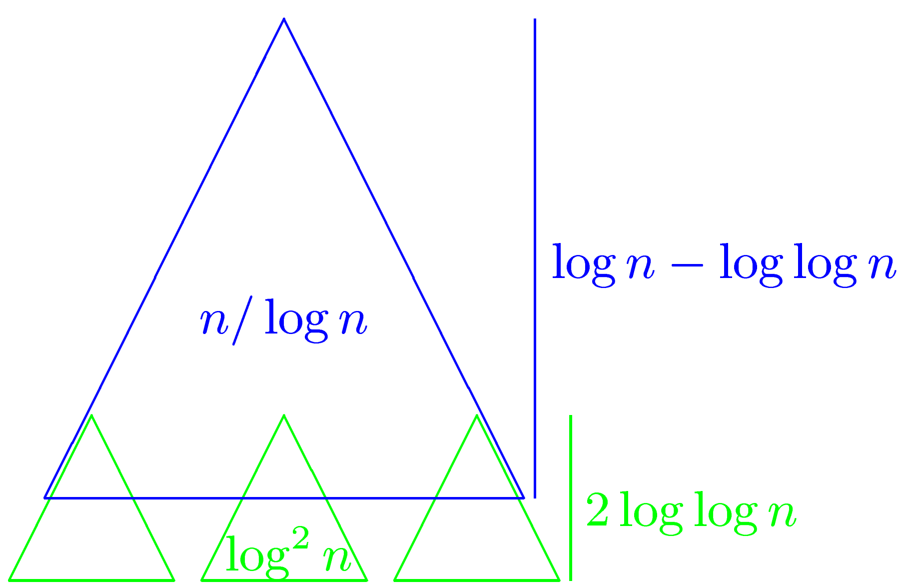
– Jimbo, Maruoka 1996

Corollary $S(n, t) = \mathcal{O}(n \log t)$

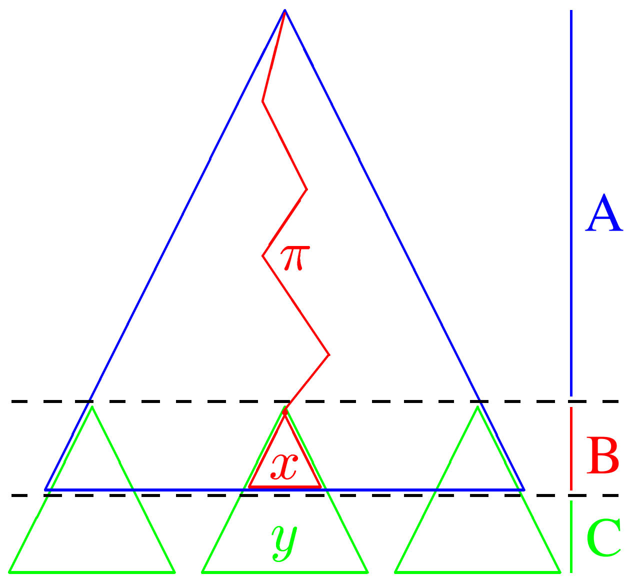
Theorem $H(n) = \mathcal{O}(n \log\log n)$

Proof of Theorem

1. Apply $(\log^2 n, \log n)$-selection networks to each green subtree $\mathcal{O}(n \log\log n)$
2. Apply $\mathcal{O}(n \log n)$-heap construction network to the blue subtree $\mathcal{O}(n)$
3. Apply $\mathcal{O}(n \log n)$-heap construction network to each green subtree $\mathcal{O}(n \log\log n)$



$\mathcal{O}(n \log \log n)$ **Heap Construction Networks**



i) A is heap-ordered after Step 2

ii) BC is heap-ordered after Step 3

In the $\mathcal{O}(n \log n)$-heap construction networks tree edges $\equiv$ comparators $\Rightarrow$ elements in $\pi \cup x$ can only be replaced by smaller elements in Step 2

For Step 2

$$x \leq y \quad \wedge \quad |\pi| \leq |x| \quad \Rightarrow \quad \pi' \leq x' \cup y$$

iii) ABC is heap-ordered after Step 3

An $\Omega(n \log \log n)$ Lower Bound

Lemma $S(n,t) \geq (n-t)\lceil \log(t+1) \rceil$

– Alekseev 1969

Theorem $H(n) \geq S(n, \lfloor \log n \rfloor) - \mathcal{O}(n)$

Corollary $H(n) \geq n \log \log n - \mathcal{O}(n)$

Proof of Theorem

Idea Reduce $(n, \lfloor \log n \rfloor)$-selection networks to heap construction networks

1. Construct a heap $H(n)$
2. Iteratively *delete* the $\lfloor \log n \rfloor$ smallest elements

   To find the t smallest elements only the t topmost levels of the heap need to be considered

   Sufficient to apply the linear size SiftDown networks to heaps of size $n, n/2, n/4, n/8, \ldots$

   Total size of the SiftDown networks

$$\sum_{i=0}^{\log n} \frac{n}{2^i} = \mathcal{O}(n)$$

Conclusion

- Heap construction networks with size $\mathcal{O}(n \log\log n)$ and depth $\mathcal{O}(\log n)$
- An $\Omega(n \log\log n)$ lower bound for the size of heap construction networks

Theorem If for constants C_1 and C_2

$$C_1 n \log t - \mathcal{O}(n) \;\leq\; S(n,t) \;\leq\; C_2 n \log t + \mathcal{O}(n)$$

then

$$C_1 n \log\log n - \mathcal{O}(n) \;\leq\; H(n) \;\leq\; C_2 n \log\log n + \mathcal{O}(n \log\log\log n)$$