The Randomized Complexity of Maintaining the Minimum

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The FindMin problem

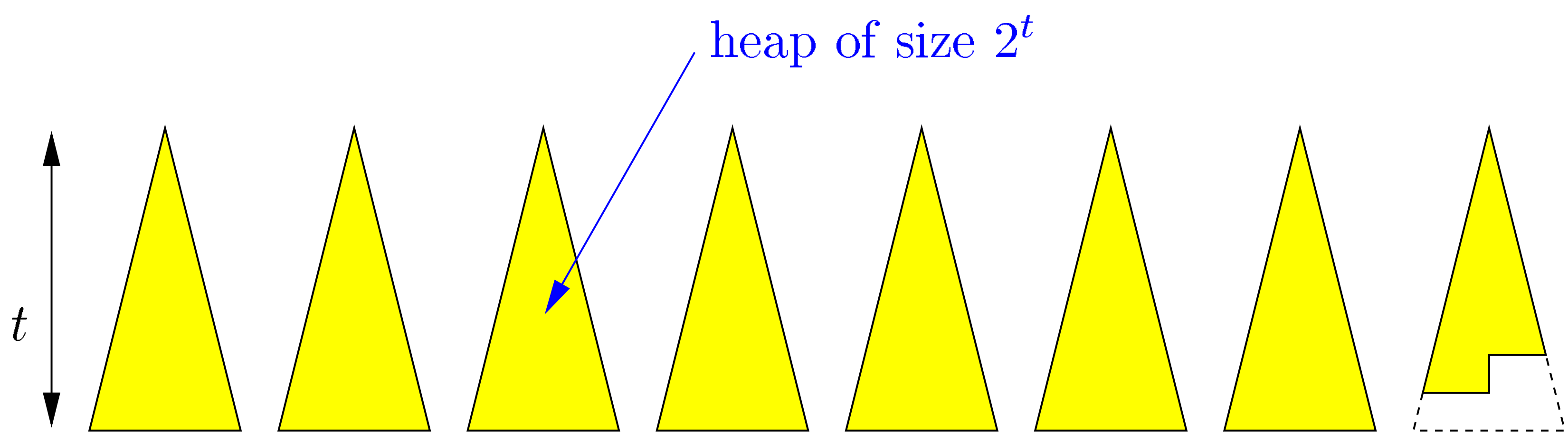
- Ordered universe $\{a_1, \ldots, a_n\}$.
- Data structure:
  - Ins, insert an element,
  - Del, delete an element (given a pointer),
  - FindMin, return the current minimum.
- Input: Sequence of operations
  - Ex.: Ins(a_1)Ins(a_5)Ins(a_8)Del(a_5)FindMin,
  - on-line,
  - comparison model,
  - worst-case cost.

Different solutions

	INS	DEL	FINDMIN
Double linked list	1	1	n
Heap	$\log n$	$\log n$	1
Search trees	$\log n$	1	1
Priority queues	1	$\log n$	1
By symmetry ?	1	1	$\log n$

Sorting $\Rightarrow$ one of the operations must cost $\Omega(\log n)$.

A simple data structure



- INS and DEL cost $O(t)$
- FINDMIN costs $O(\frac{n}{2^t})$

The FindAny problem

- Data structure:
  - Ins, Del,
  - FindAny, return an arbitrary element and its current rank.
- Not harder than the FindMin problem.
- Sorting $\Rightarrow$ one of the operations must cost $\Omega(\log n)$.

Lower bounds

Theorem. For any deterministic data structure with cost for INS and DEL at most t, the cost of FINDANY is

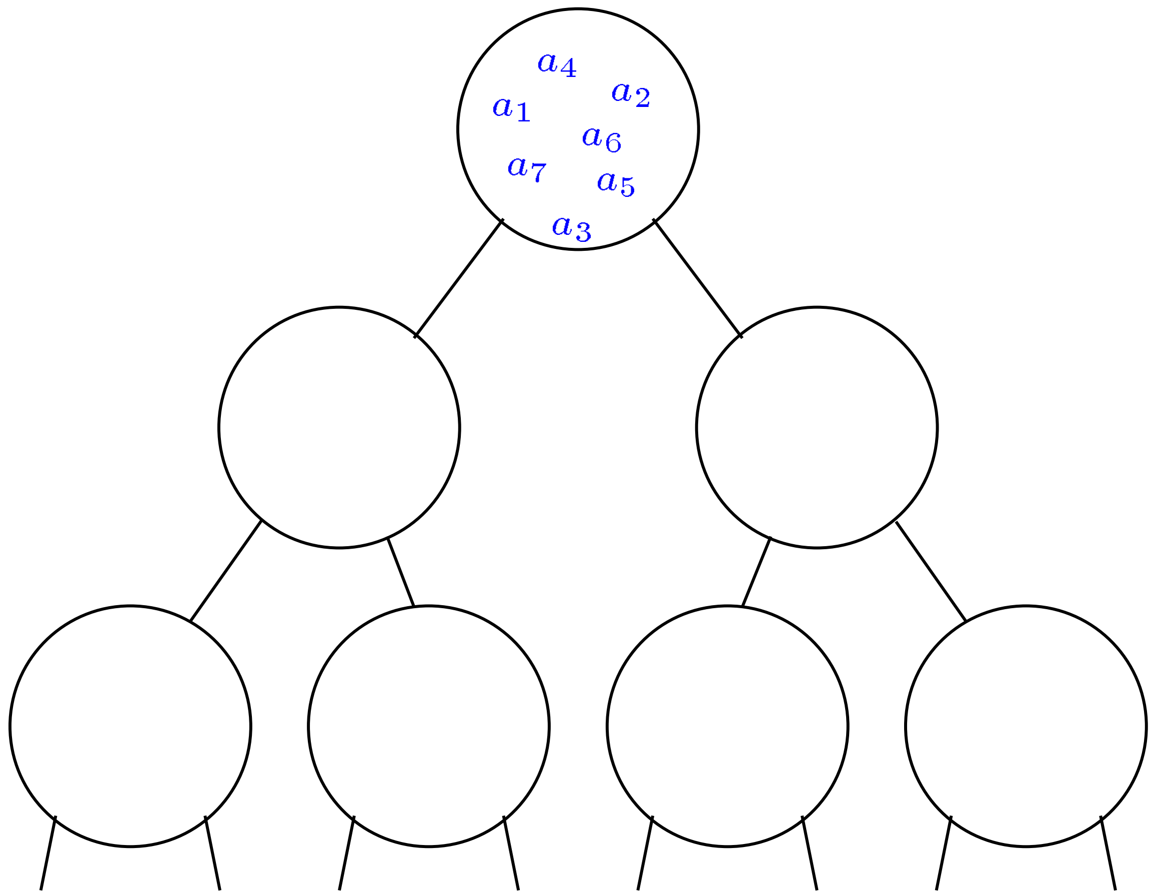
$$\geq \frac{n}{2^{4t+3}} - 1.$$

Theorem. For any randomized data structure with expected cost for INS and DEL at most t, the expected cost of FINDMIN is

$$\geq \frac{n}{e2^{2t}} - 1.$$

An adversary strategy for FindAny

Infinite binary tree. Initially, all elements are in the root.



- Adversary answers consistently.
- If v, w are on the same path from the root, then v, w are incomparable in the algorithms poset.
- A comparison pushes ≤ 2 elements one level down.
- Consider $\textsc{Ins}(a_1) \cdots \textsc{Ins}(a_n)\textsc{Del}(b_1) \cdots \textsc{Del}(b_n)$ where b_i is the element that would have been returned by FindAny.
- Assign b_i to the node where it is deleted.
- If k elements are deleted at some node and b_m the first among them, then replacing $\textsc{Del}(b_m)$ by FindAny takes at least $k-1$ comparisons.
- Sum of the depths of $b_1, \ldots, b_n$ is $\leq 2 \cdot 2tn = 4tn$
  $\Downarrow$
  there exists a node where $\dfrac{n}{2^{4t+3}}$ elements were deleted.

An explicit sorting adversary

When the elements are sorted, each element must lie in a distinct leaf.

$$\Downarrow$$

Sum of the depths of the nodes $\geq n \log n$.

$$\Downarrow$$

Number of comparisons $\geq \dfrac{n \log n}{2}$.