Cache-Oblivious Search Trees via Trees of Small Height

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BRICS

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Joint work with Rolf Fagerberg and Riko Jacob

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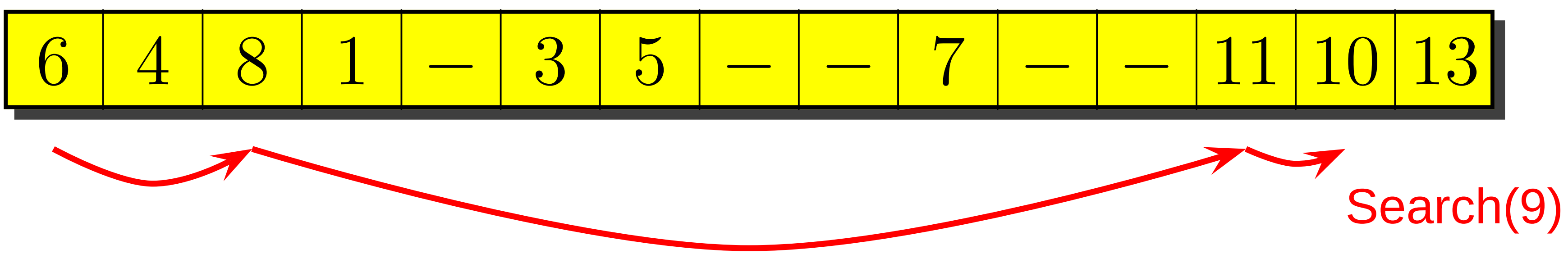
New Search Tree

$$\{\ 1,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 10,\ 11,\ 13\ \}$$

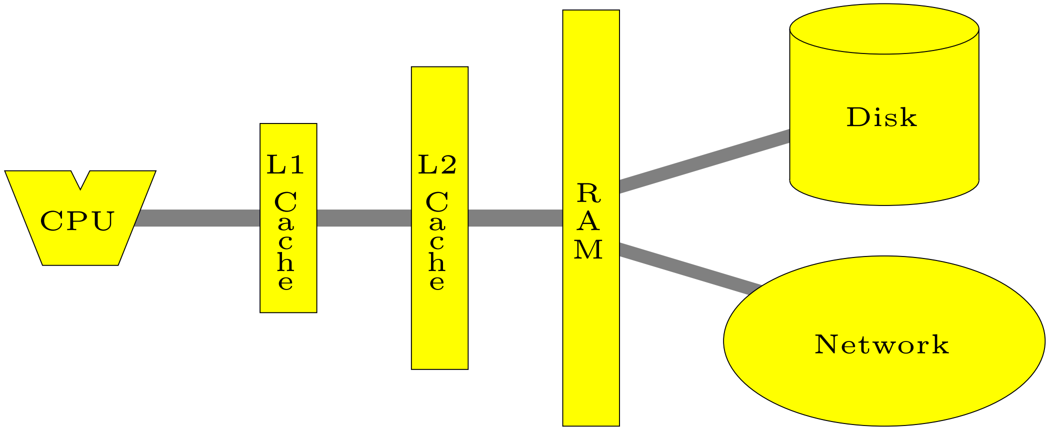
$$\Downarrow$$



Outline

- Trends in Implementation Technology
- Models of Computation
  - I/O Model
  - Cache-Oblivious Model
- Cache-Oblivious Search Trees
  - Static
  - Dynamic
- Experiments
  - Memory Layouts of Trees
- Summary

Trends in Implementation Technology



Integrated Circuit Logic Technology

- Transistor count increases $\approx$ 60-80% per year

DRAM

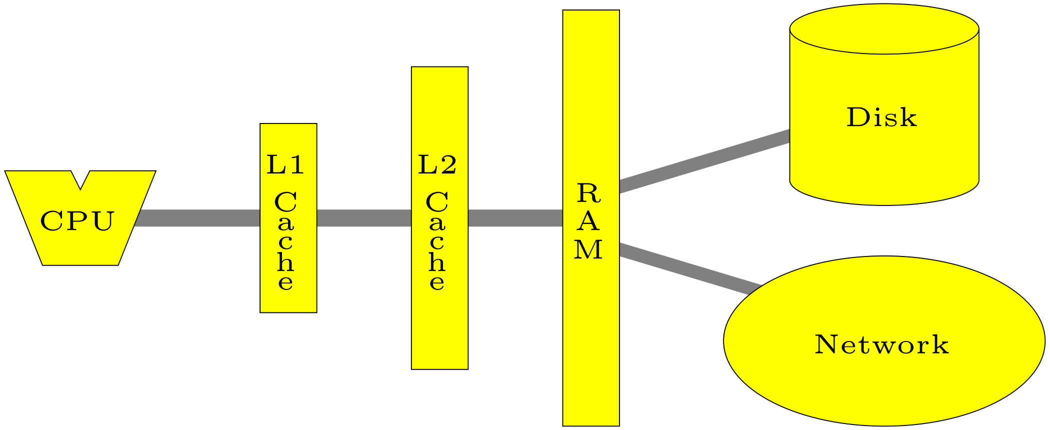
- Density improves $\approx$ 60% per year
- Cycle time improves $\approx$ 35% per 10 years

Magnetic Disk

- Density improves $\approx$ 50% per year
- Access time improves $\approx$ 35% per 10 years

Source: *Computer Architecture – A Quantitative Approach*, Hennessy & Patterson, 2nd. Ed. 1996

Trends in Implementation Technology



	L1 Cache	L2 Cache	Virtual memory
Block size	4 – 32 bytes	32 – 256 bytes	4 – 16 KB
Hit time (cycles)	1 – 2	6 – 15	10 – 100
Miss penalty (cycles)	8 – 66	30 – 200	700.000 – 6.000.000
Size	1 – 128 KB	256 KB – 16 MB	16 – 8192 MB

Source: *Computer Architecture – A Quantitative Approach*, Hennessy & Patterson, 2nd. Ed. 1996

The Unknown Machine

Algorithm

↓

C program

↓ gcc

Object code

↓ linux

Execution

Can be executed on machines with a specific class of CPUs

Algorithm

↓

Java program

↓ javac

Java bytecode

↓ java

Interpretation

Can be executed on any machine with a Java interpreter

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C program	Java program
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Goal Develop algorithms that are optimized w.r.t. memory hierarchies without knowing the parameters

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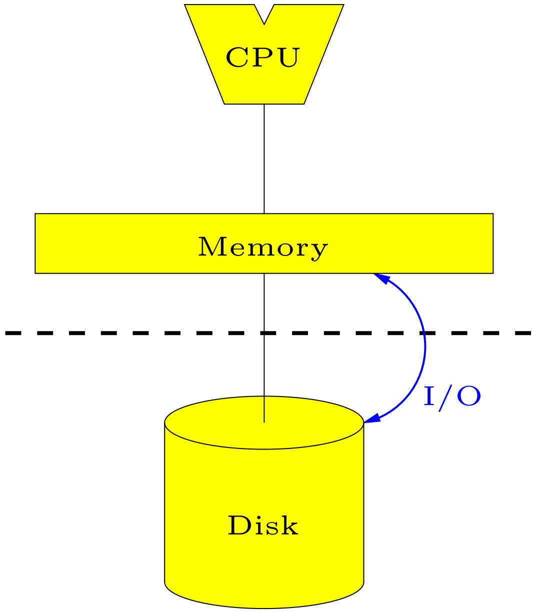
I/O Model



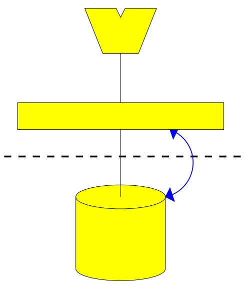
N = problem size
M = memory size
B = I/O block size

Aggarwal and Vitter 1988

- Bottleneck $\equiv$ I/Os between the two highest memory levels
- B-trees support searches and updates in $O(\log_B N)$ I/Os
- $\Theta\left(\frac{M}{B}\right)$-way merge-sort achieves optimal $\Theta\left(\frac{N}{B}\log_{M/B}\frac{N}{B}\right)$ I/Os

Cache-Oblivious Model

- I/O model
- Algorithms do not know the parameters B and M
- Optimal off-line cache replacement strategy



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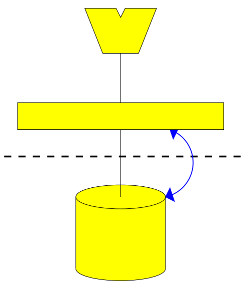
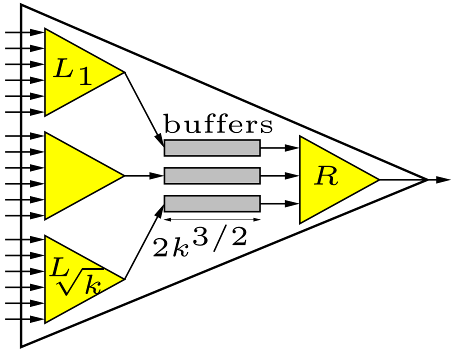


Examples

- Scanning, Linear time selection
- Matrix-transposition, FFT, Funnel-sorting



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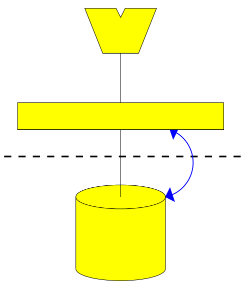

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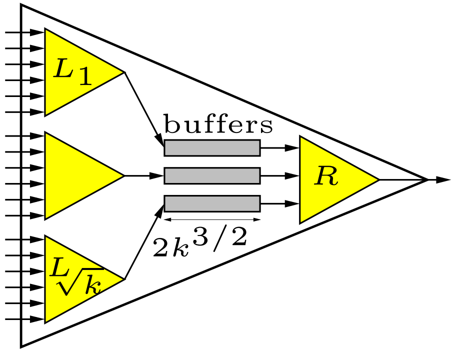

Lemma

Optimal cache-oblivious algorithm implies optimal algorithm on each level of a fully associative multi-level cache using LRU

Frigo et al. 1999

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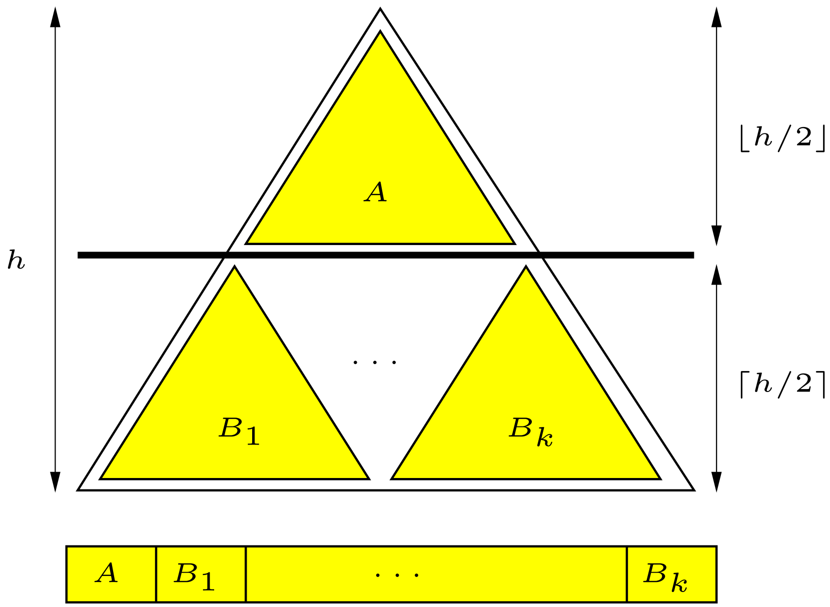
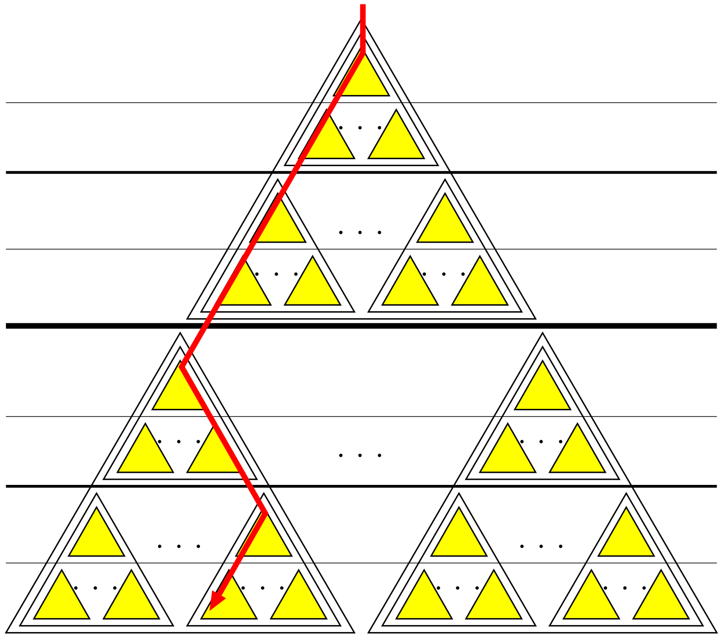
Static Cache-Oblivious Trees

Recursive memory layout $\equiv$ van Emde Boas layout



Binary tree



Searches use $O(\log_B N)$ I/Os

Prokop 1999

Static Cache-Oblivious Trees

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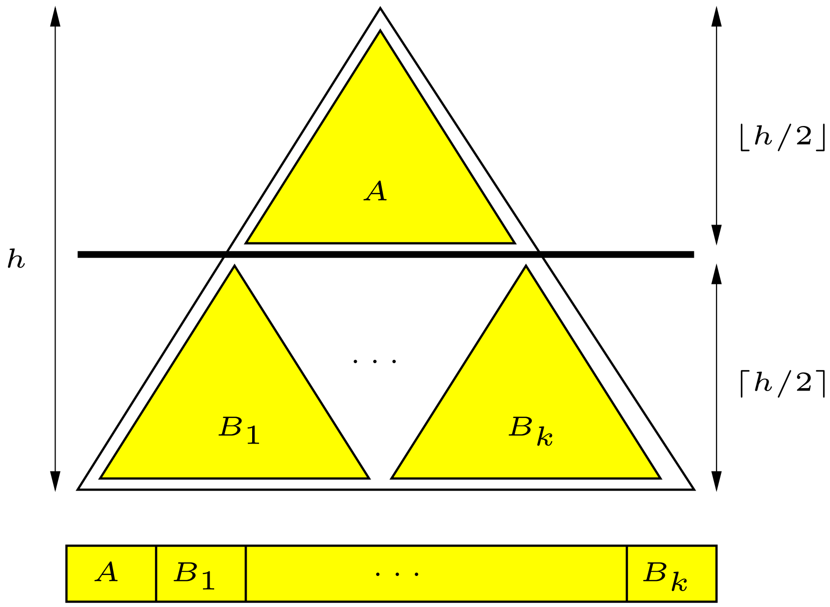

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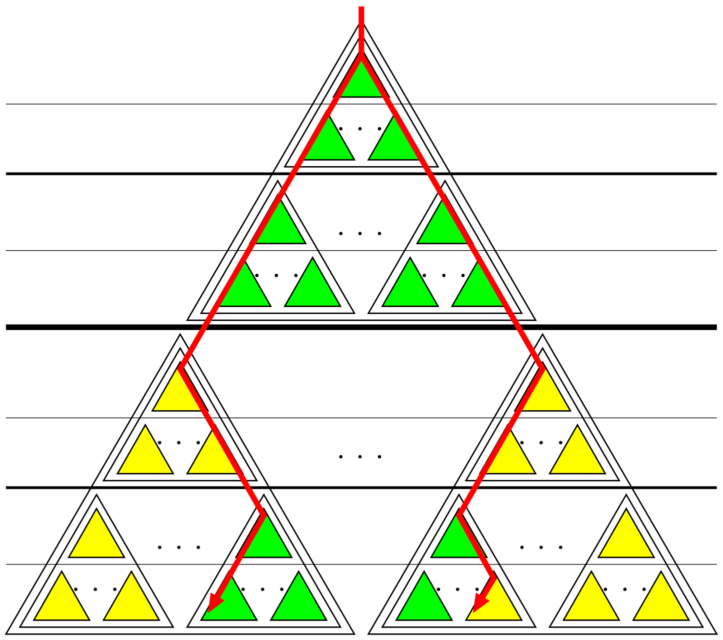

Searches use $O(\log_B N)$ I/Os

Range reportings use

$O\left(\log_B N + \frac{k}{B}\right)$ I/Os

Prokop 1999

Dynamic Cache-Oblivious Trees

Search	$\mathrm{O}(\log_B N)$
Range Reporting	$\mathrm{O}\left(\log_B N + \frac{k}{B}\right)$
Updates	$\mathrm{O}\left(\log_B N + \frac{\log^2 N}{B}\right)$

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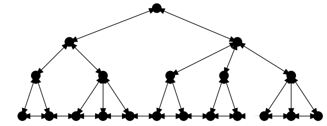
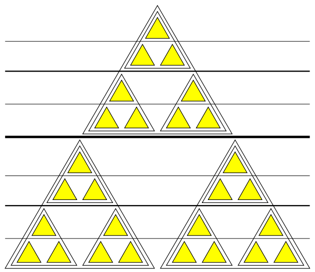
- Pointer Based Strongly Weight Balanced B-trees
- Dynamic van Emde Boas Layout
- Packed Memory Management
- Two Levels of Indirection



Arge and Vitter 1996



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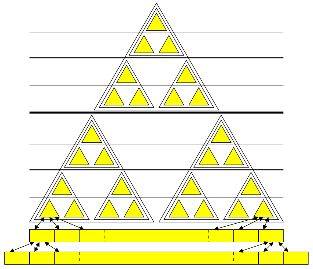
Itai et al. 1981



Bender, Demain, Farach-Colton 2000

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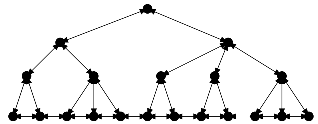

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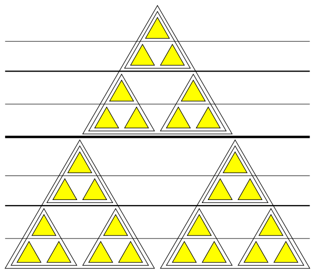

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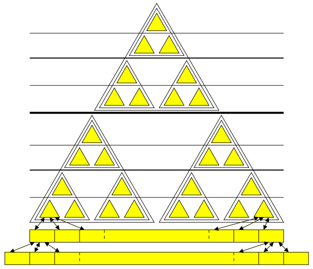

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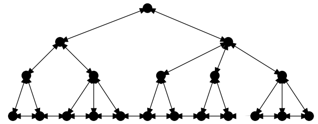

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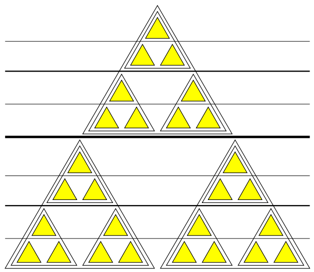

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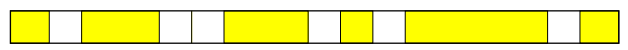

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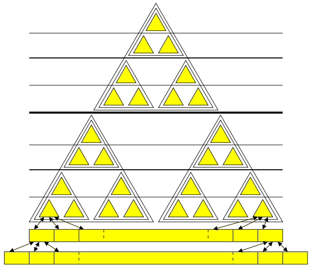

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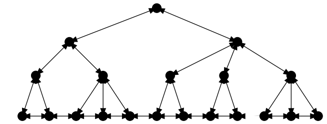

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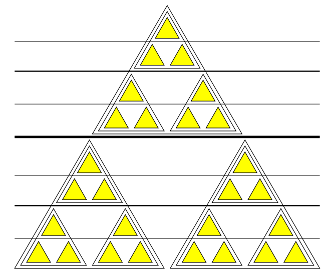

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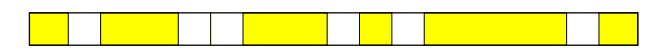

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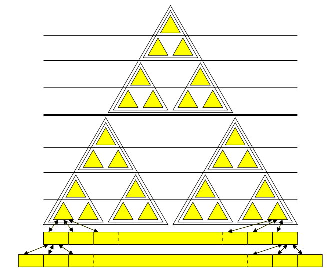

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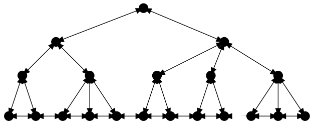

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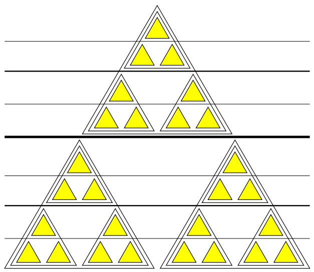

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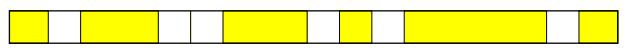

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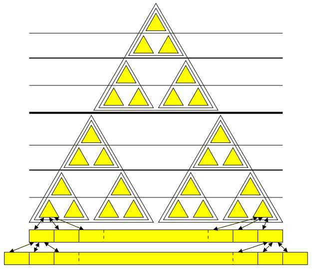

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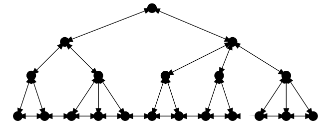

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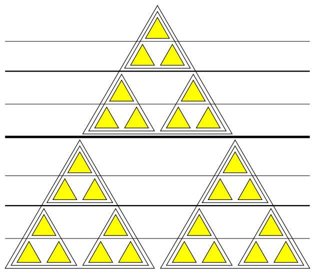

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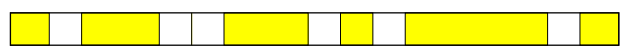

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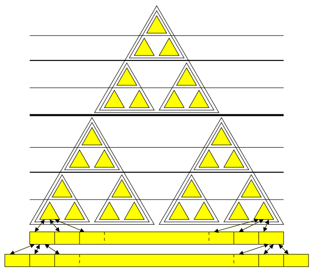

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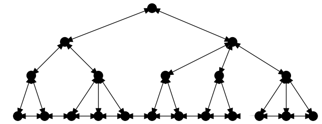

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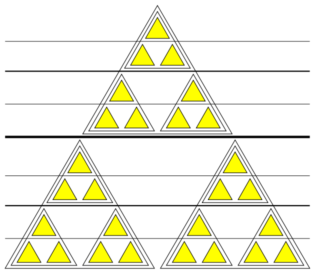

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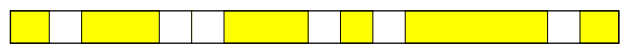

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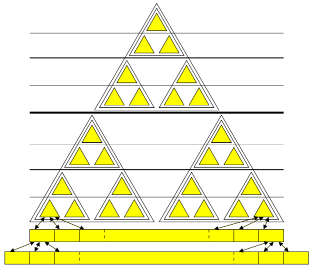

- Pointer Based Strongly Weight Balanced B-trees
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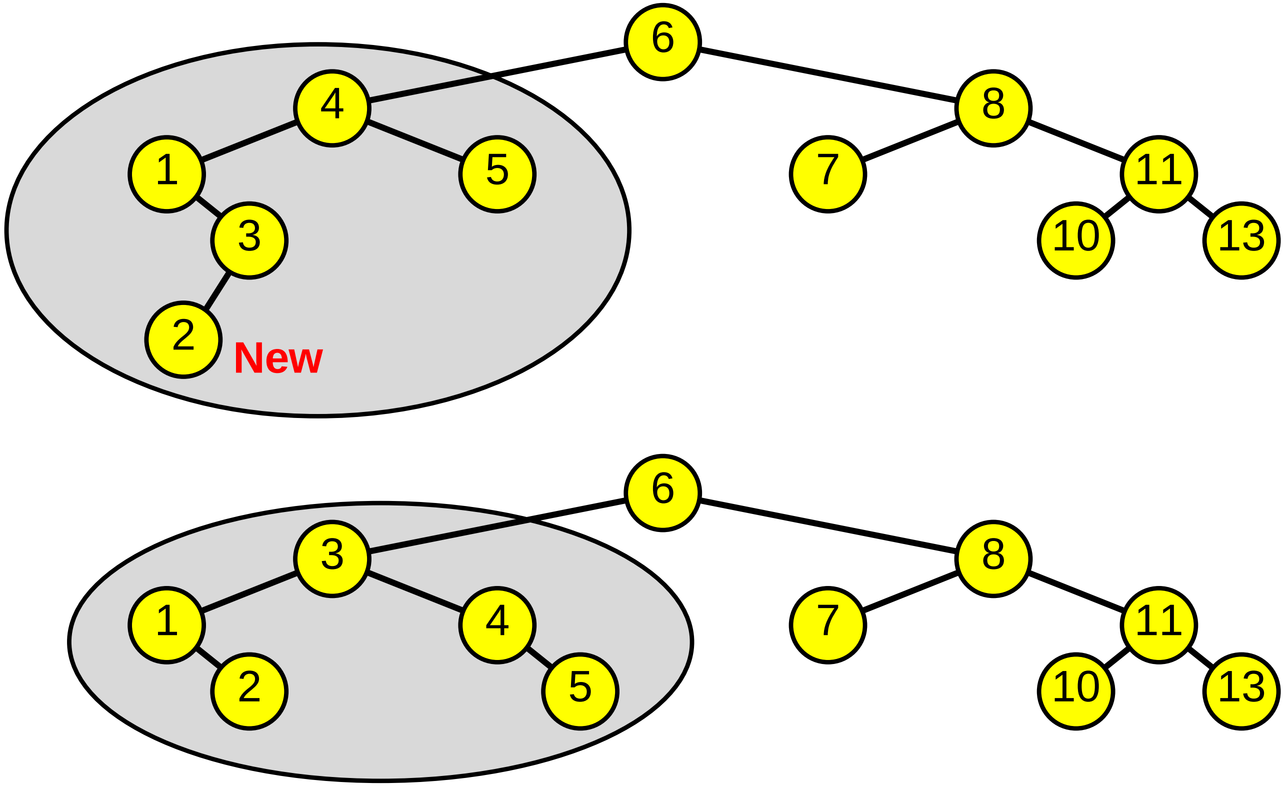
Bender, Demain, Farach-Colton 2000

Binary Trees of Small Height

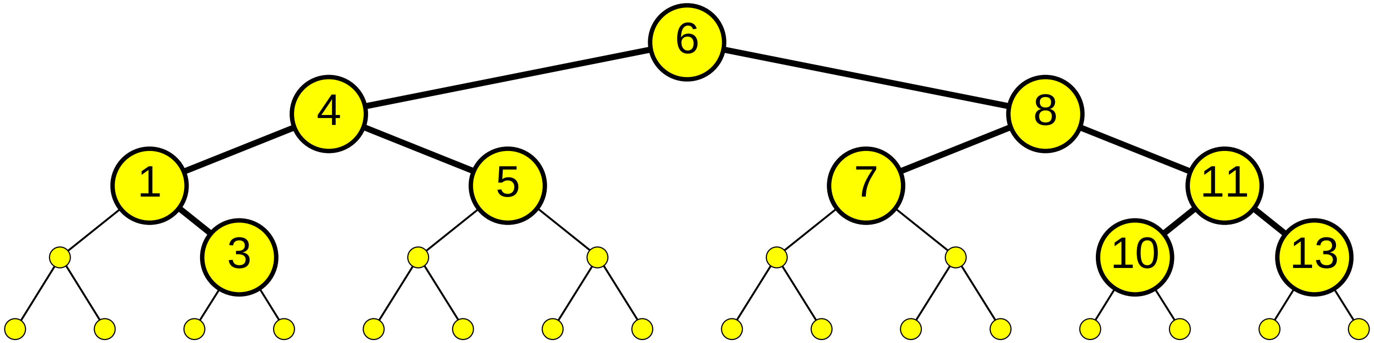


- If an insertion causes non-small height then rebuild subtree at nearest ancestor with sufficient few descendents
- Insertions require amortized time $\mathrm{O}(\log^2 N)$

Andersson and Lai 1990

Dynamic Cache-Oblivious Trees

- Embed a dynamic tree of small height into a complete tree
- Static van Emde Boas layout



Search	$O(\log_B N)$
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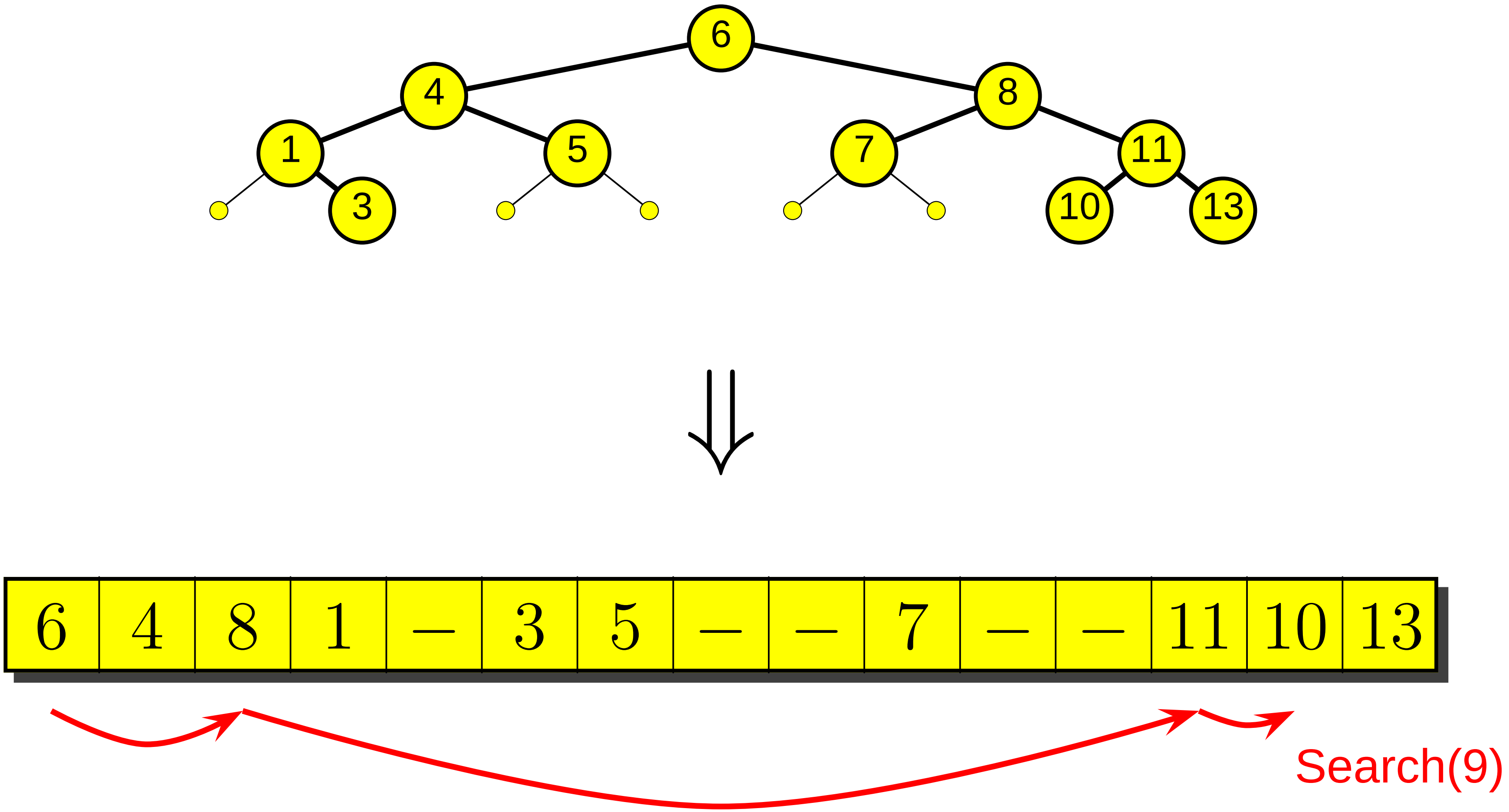
New

Example



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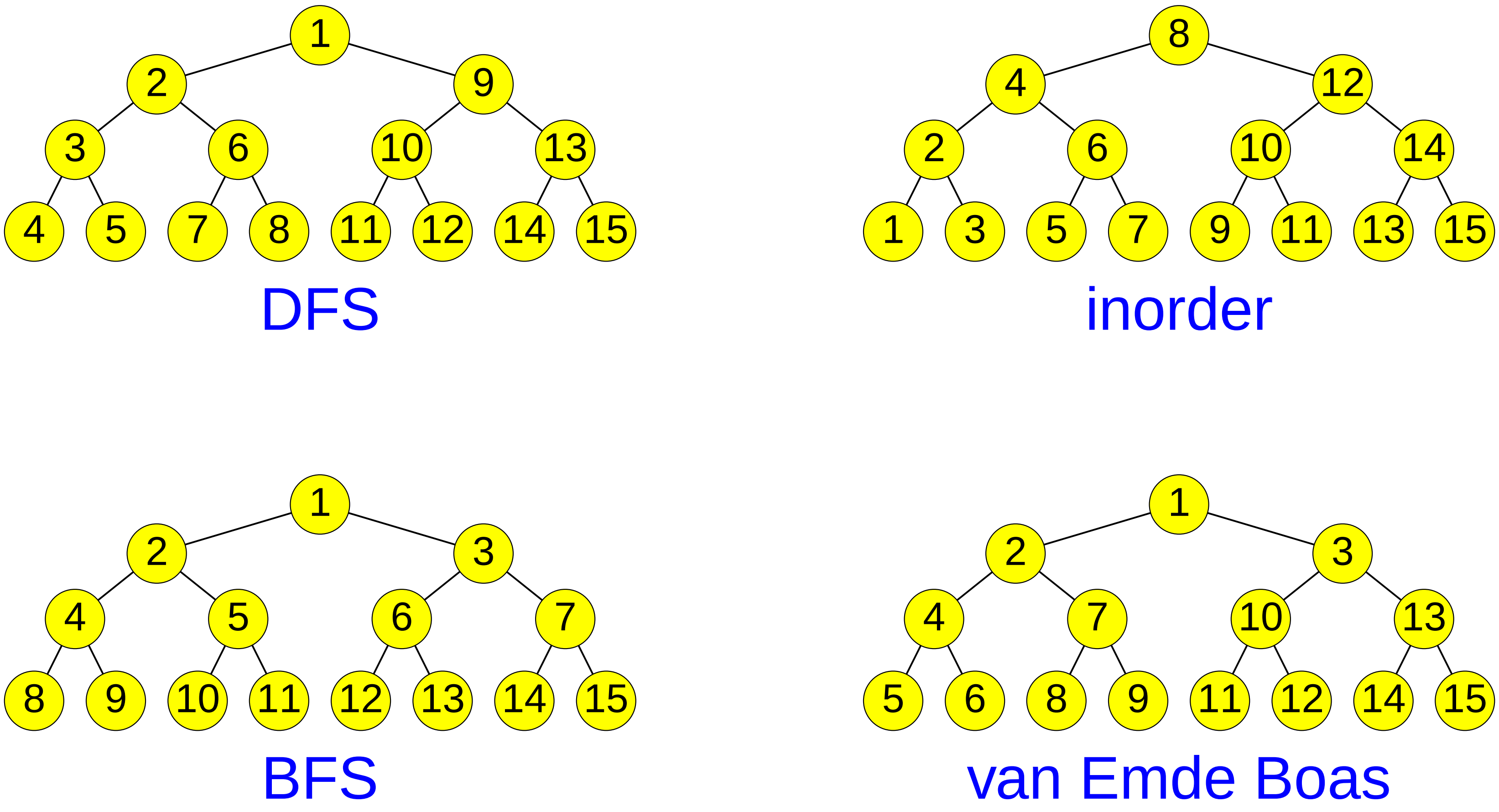
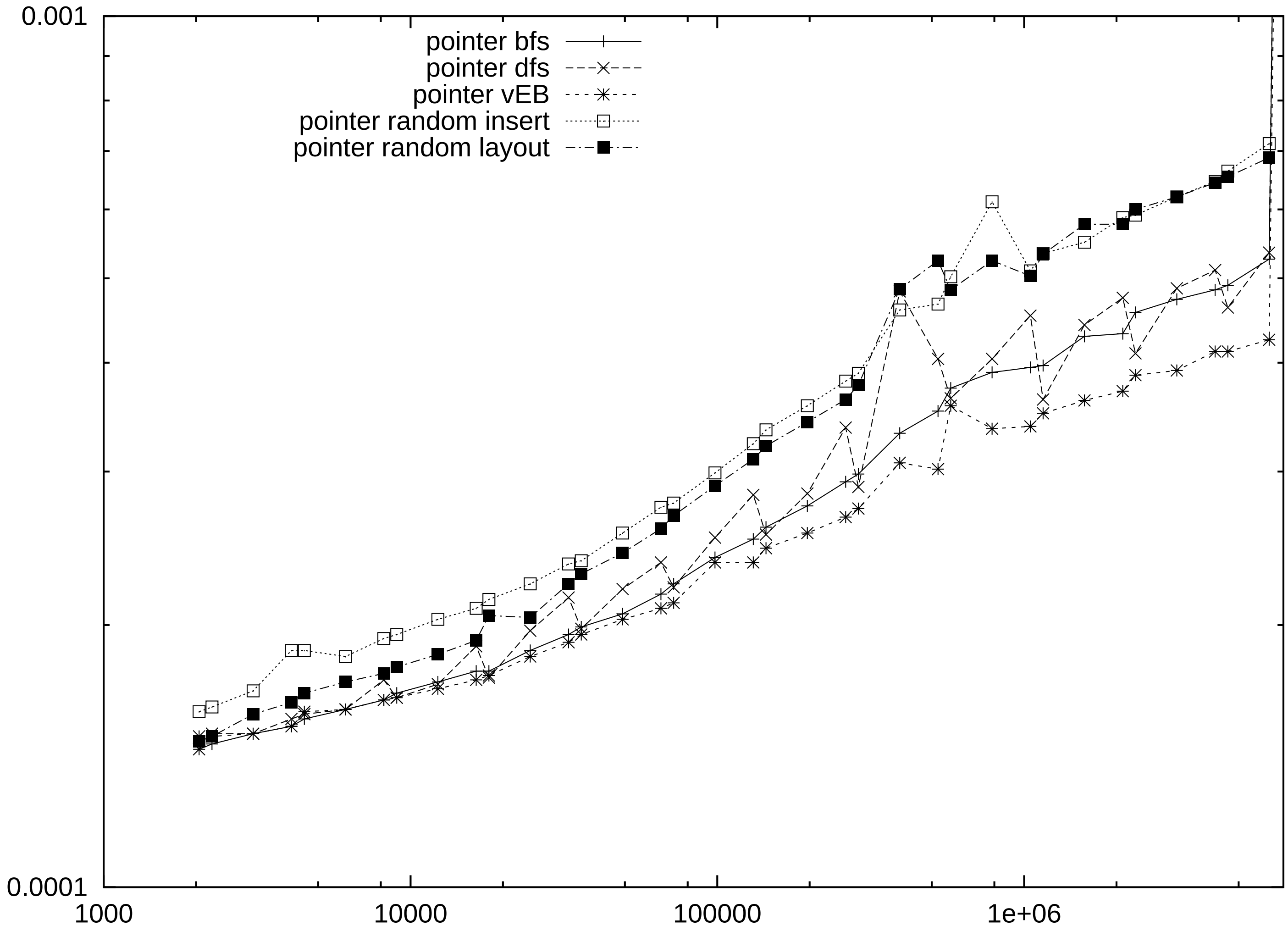
Memory Layouts of Trees

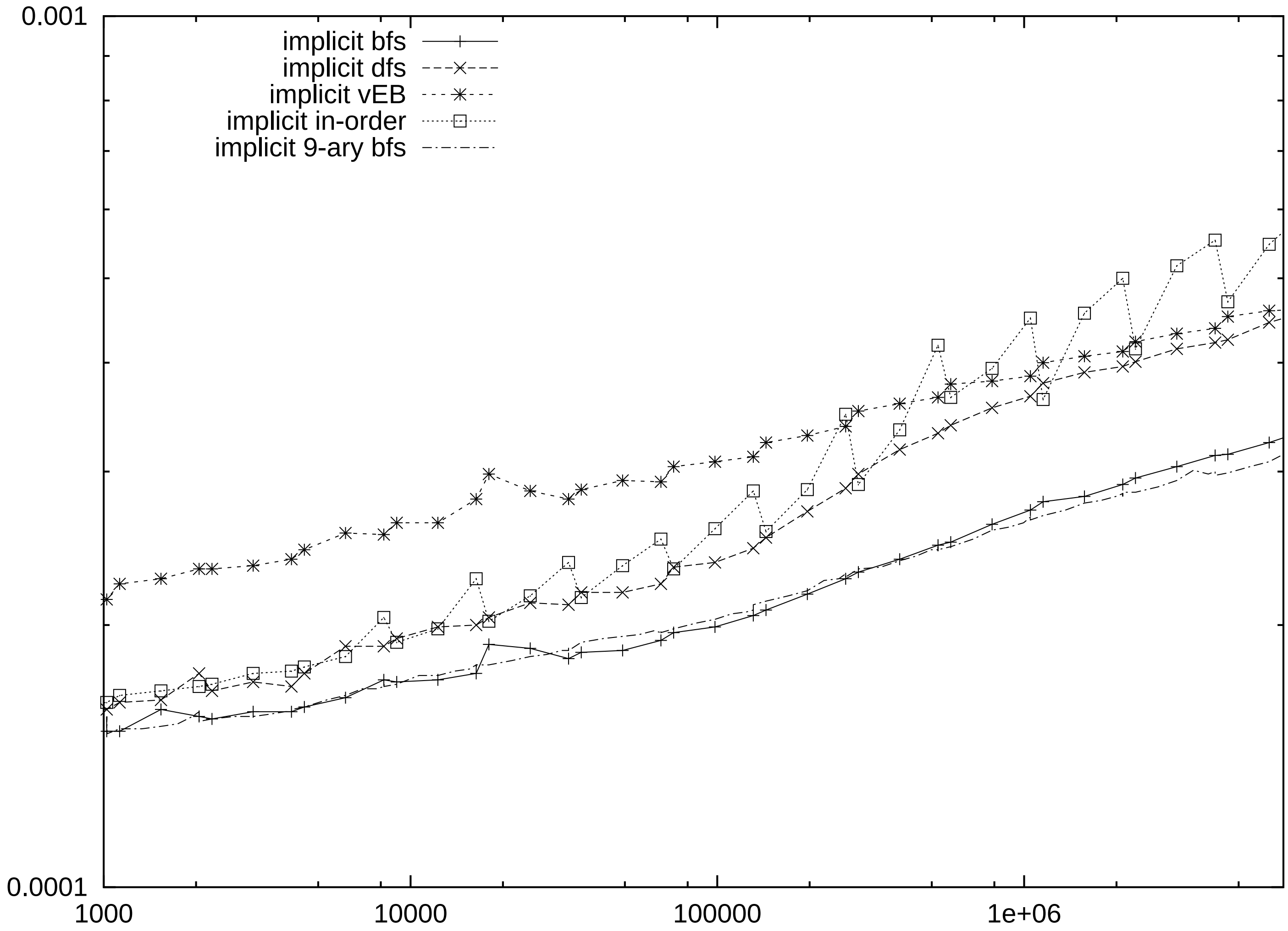


Searches in Pointer Based Layouts



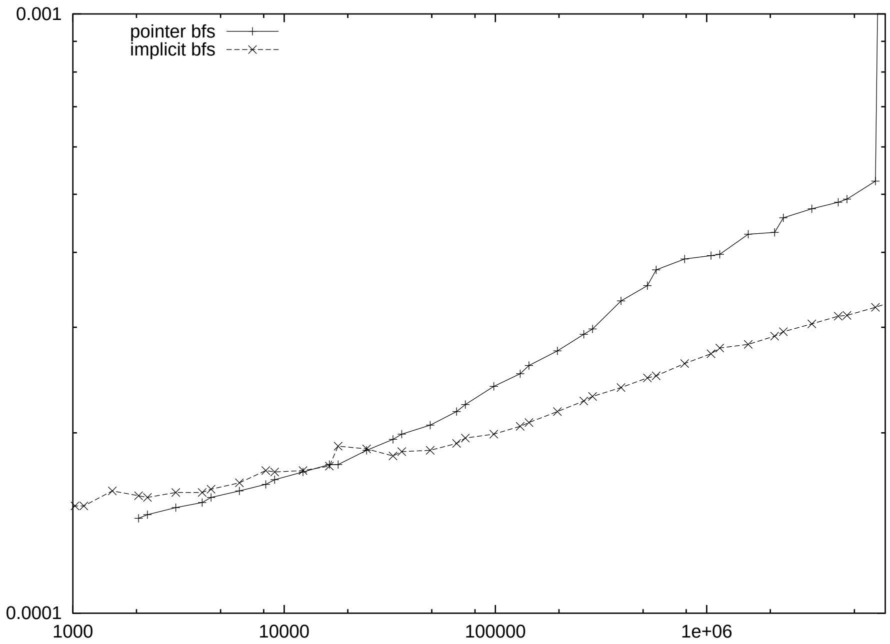
- van Emde Boas layout wins, followed by the BFS layout

Searches with Implicit Layouts

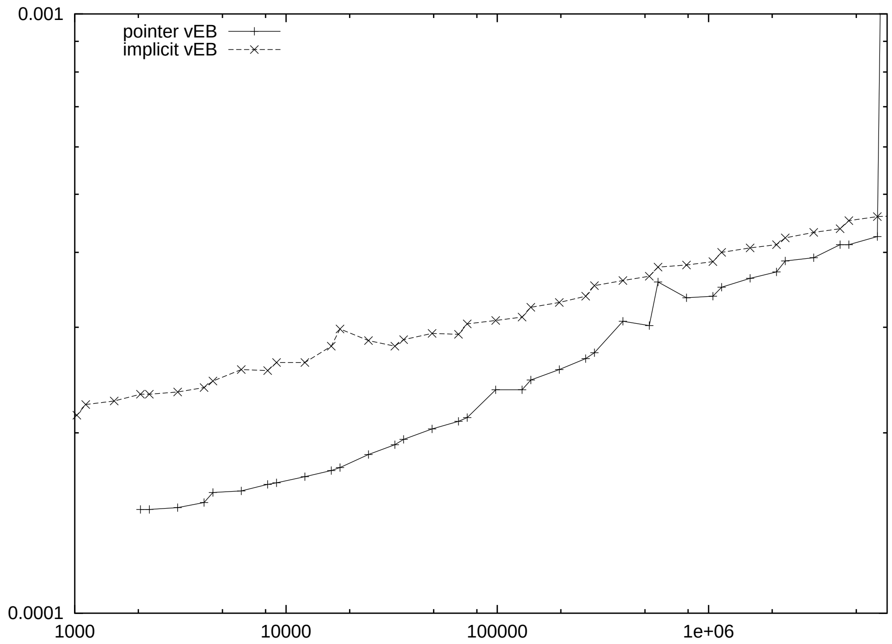


- BFS layout wins due to simplicity and caching of topmost levels
- van Emde Boas layout requires quite complex index computations

Implicit vs Pointer Based Layouts



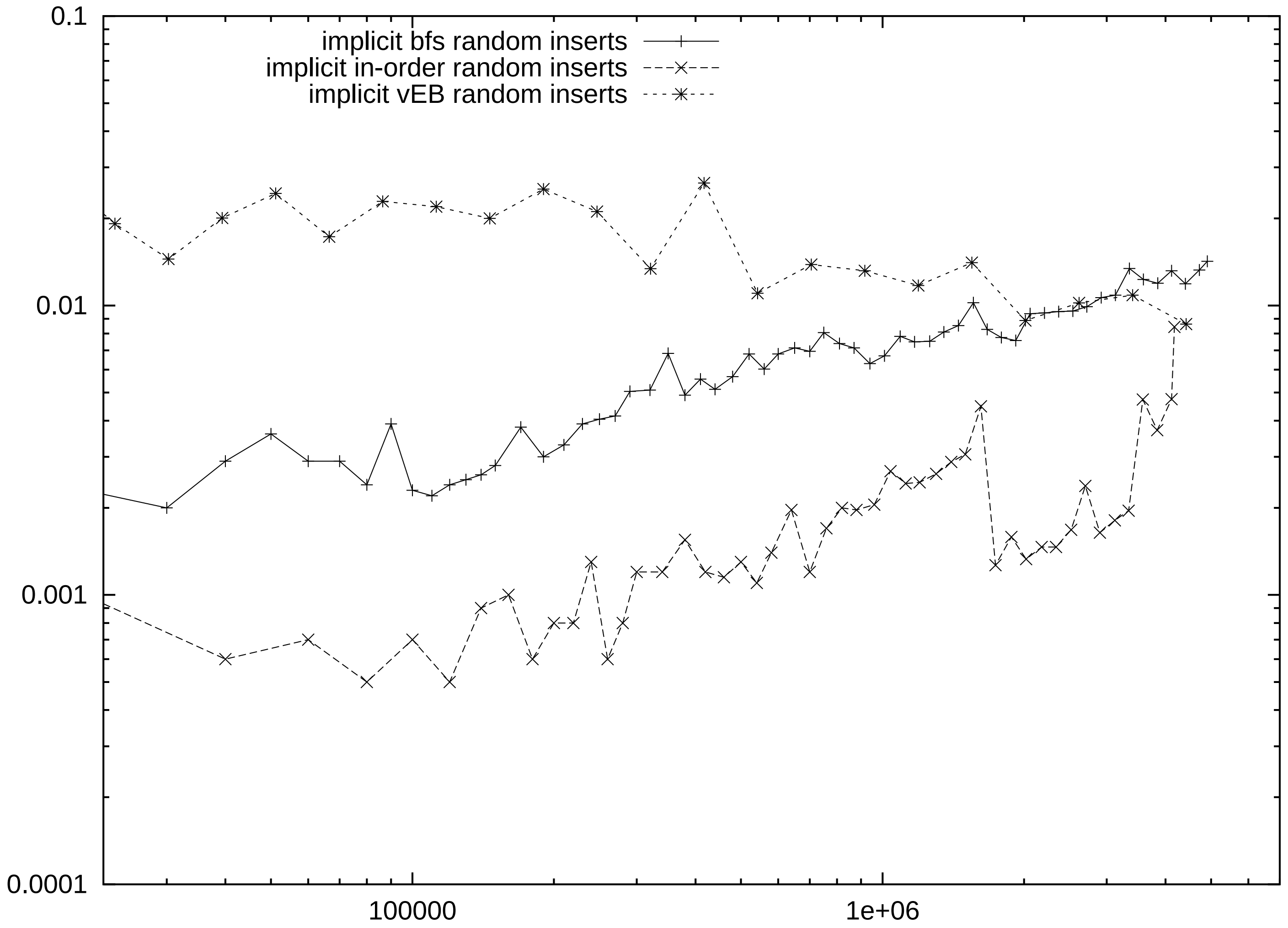
BFS layout



van Emde Boas layout

- Implicit layouts become competitive as n grows

Insertions in Implicit Layouts



- Insertions are rather slow (factor 10-100 over searches)

Summary

- Simple cache-oblivious search trees

Search	$O(\log_B N)$
Range Reporting	$O\left(\log_B N + \frac{k}{B}\right)$
Updates	$O\left(\log_B N + \frac{\log^2 N}{B}\right)$

- Importance of memory layouts
- van Emde Boas layout gives good cache performance
- Computation time is important when considering caches
- Update time $O(\log_B N)$ by one level of indirection (implies sub-optimal range reporting)