Unified Access Bound


- Dictionary: Insert(x), Delete(x), Search(x)
- Comparison model

Solution 1: Balanced search tree \(\Rightarrow O(\log n)\)

Solution 2: Unordered linked list \(\Rightarrow O(n)\)

The paper initiated the study of competitiveness analysis of online algorithms for list ordering, search-trees, paging algorithms, ... (move-to-front is 2-competitive)

Access sequences - examples

- $X_1 = 1, 2, 3, \ldots, n, 1, 2, 3, \ldots, n, 1, 2, 3, \ldots$
- $X_2 = 1, n, 1, n, 1, n, \ldots$
- $X_3 = 1, n/2, 2, n/2+1, 3, n/2+2, \ldots, n/2, n, 1, \ldots$
Access sequence $X = (x_1, x_2, \ldots, x_m)$

- Static optimal: $O(\log (1/p(x_i)))$
- Sequential-access bound: $O(1)$
- Static finger bound: $O(\log d_i(f,x_i))$
- Dynamic finger bound: $O(\log d_i(x_i, x_{i-1}))$
- Working set bound: $O(\log w_i(x_i))$
- Unified bound: $O(\min_{y \in S_i} \log(w_i(y) + d_i(x_i, y)))$
Splay trees (amortized)

★ 1 Static optimal  O(\log (1/p(x_i)))
★ 2 Sequential-access bound  O(1)
★ 1 Static finger bound  O(\log d_i(f,x_i))
★ 3 Dynamic finger bound  O(\log d_i(x_i, x_{i-1}))
★ 1 Working set bound  O(\log w_i(x_i))
Open
■ Unified bound  O(\min_{y \in S_i} w_i(y) + d_i(x_i, y))

Static optimality


Split $\leq \frac{1}{2}$ weight both children
$\Rightarrow$ Depth $i$ subtree weight $\leq (\frac{1}{2})^i$
$\Rightarrow$ Depth $x_i \leq \log(1/p(x_i))$
$\Rightarrow$ Static optimal

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td>p($x_i$)</td>
<td>0.10</td>
<td>0.05</td>
<td>0.02</td>
<td>0.60</td>
<td>0.03</td>
<td>0.08</td>
<td>0.02</td>
<td>0.06</td>
<td>0.01</td>
<td>0.03</td>
</tr>
</tbody>
</table>

Construction: Compute prefix sums + Exponential search $\Rightarrow O(n)$
Working-set structure


- $L = L_0 + L_1 + \cdots = \text{move-to-front list}$
- $|L_i| = 2^{2^i}$
- $T_i = \text{search tree over } L_i$
- Insert, Delete $= O(\log n)$, Search $= O(\log w_j)$
Unified structure


Search trees of a **subset** of the elements

Finger search tree over **all** elements

**Lemma 7** \( w_f(y) \leq 2^k \) and \( x \) and \( y \) rank distance \( \leq 2^k \), then \( x \) within rank distance \( (k+4)2^k \) of some \( y' \in T_0 \cup \cdots \cup T_k \)