Modal Functional Interpretation

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Abstract. We extend Gödel's Dialectica interpretation to modal formulas and prove it sound for a certain modal arithmetic based on Gödel's T. The range of this *Modal Dialectica* interpretation is the usual Heyting Arithmetic in all finite types HA^ω. We illustrate the use of the new tool for optimized program extraction as part of an enhanced light Dialectica interpretation in the sense of [6].

This note comes in completion of paper [6] by adding a device for summing the previous optimizations by semi- and non-computational quantifiers in a compact one-step *content eraser*, the modal operator $\Box$ (and its co-modality $\Diamond$). Besides the "cosmetic" improvement, we bring the following new result: while the modal propositional axioms of system S_4 are realizable, the defining axiom of S_5 is generally not realizable.

1 Arithmetical systems for Modal Dialectica extraction

We build upon systems NA and NA_l from [6]. System MA is obtained simply by adding to NA the strong existential quantifier $\exists$ via the usual axioms (see [3,11]). The following are added to NA_l in order to obtain MA_l:

- the strong existentials $\exists_\pm$ and $\exists_\emptyset$, which are the $\exists$ and respectively $\overline{\exists}$ from [3];
- the well-known modal operators $\Box$ for necessity and $\Diamond$ for possibility; we will later describe the use of such modalities for optimizing program extraction under Dialectica interpretation.

With the addition of $\Box$, $\Diamond$ and respectively $\exists$, $\exists_\emptyset$, $\exists_\pm$, the sets of finite types T, terms $\mathcal{T}$ and formulas $\mathcal{F}/\mathcal{F}_l$ are defined as follows (for simplifying notation, we can use $\forall$ for $\forall_\pm$ and $\exists$ for $\exists_\pm$):

$$
\begin{array}{llcl}
T & \rho,\sigma & ::= & \iota \mid o \mid (\rho\sigma) \\
\mathcal{T} & s,t & ::= & x^\rho \mid \mathtt{T}^o \mid \mathtt{F}^o \mid \mathtt{0}^\iota \mid \mathtt{S}^{\iota\iota} \mid \mathtt{If}^{o\rho\rho\rho} \mid \mathtt{R}^{\iota\rho(\iota\rho\rho)\rho} \mid (\lambda x^\rho.\, t^\sigma)^{\rho\sigma} \mid (t^{\rho\sigma} s^\rho)^\sigma \\
\mathcal{F} & A,B & ::= & \mathtt{at}(t^o) \mid A \to B \mid A \wedge B \mid \forall x^\rho A \mid \exists x^\rho A \\
\mathcal{F}_l & A,B & ::= & \mathtt{at}(t^o) \mid A \to B \mid A \wedge B \mid \forall_{\{\emptyset,+,-,\pm\}} x^\rho A \mid \exists_{\{\emptyset,\pm\}} x^\rho A \mid \Box A \mid \Diamond A
\end{array}
$$

Recall that we employ just two basic types: integers ι and booleans o, and use $\rho\sigma\tau$ for $(\rho(\sigma\tau))$. Building blocks for terms are the usual constructors for booleans $(\mathtt{T},\mathtt{F})$ and integers $(\mathtt{0},\mathtt{S})$, case distinction $\mathtt{If}$ and Gödel recursion $\mathtt{R}$.

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The operator $\mathsf{FV}(\cdot)$ returns the set of free variables of its argument $t \in \mathcal{T}$ or $A \in \mathcal{F}$. Atomic formulas are decidable by definition, as they are identified with boolean terms. In particular, we have decidable falsity $\bot :\equiv \mathtt{at}(\mathtt{F})$ and truth $\top :\equiv \mathtt{at}(\mathtt{T})$. As usual, we abbreviate $A \to \bot$ by $\neg A$.

We will not detail here the arithmetics NA and NA_l, but rather refer the reader to [6]. Below we enumerate only the items that are added in order to get MA and respectively MA_l. As already mentioned, strong existence comes with the defining axioms (identical axioms are in use for $\exists_\emptyset$, see [3]):

$\mathtt{Ax}\exists^i :\ \forall a[A(a) \to \exists z A(z)]$ and

$\mathtt{Ax}\exists^e :\ \exists z\, A(z) \wedge \forall a\, [\, A(a) \to B\,] \to B,$ where $a \notin \mathsf{FV}(B)$.

The following axioms of propositional logic S_4 are added to MA_l:

$\mathtt{AxT} :\ \Box A \to A$ $\qquad$ $\mathtt{AxT}' :\ A \to \Diamond A$

$\mathtt{Ax4} :\ \Box A \to \Box\Box A$ $\qquad$ $\mathtt{Ax4}' :\ \Diamond\Diamond A \to \Diamond A$

$\mathtt{AxK} :\ [\Box(A \to B) \wedge \Box A] \to \Box B$

For the necessity operator we have the following introduction rule, denoted $\Box^i$:

$\dfrac{\Gamma \vdash A}{\Gamma \vdash \Box A},$ where Γ is restricted depending on the translation of the (sub)proof of the premise sequent, in a way that will be described below.

We denote by $A \to_k B :\equiv \Box A \to B$ the so-called "Kreisel implication", since its translation by Modal Dialectica coincides with its Modified Realizability interpretation.

Definition 1 (The modal Dialectica interpretation). The interpretation does not change atomic formulas, i.e., $|\mathtt{at}(t^o)| :\equiv \mathtt{at}(t^o)$. Assuming $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ and $|B|^{\boldsymbol{u}}_{\boldsymbol{v}}$ are already defined,

$$\begin{aligned}
|A \wedge B|^{\boldsymbol{x},\boldsymbol{u}}_{\boldsymbol{y},\boldsymbol{v}} &:\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \wedge |B|^{\boldsymbol{u}}_{\boldsymbol{v}} \\
|A \to B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{v}} &:\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}\boldsymbol{v}} \to |B|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{v}} \\
|\forall z\, A(z)|^{\boldsymbol{f}}_{z,\boldsymbol{y}} &:\equiv |A(z)|^{\boldsymbol{f}z}_{\boldsymbol{y}} \\
|\exists z\, A(z)|^{z,\boldsymbol{f}}_{\boldsymbol{y}} &:\equiv |A(z)|^{\boldsymbol{f}}_{\boldsymbol{y}}
\end{aligned}$$

$$\begin{aligned}
|\Box A|^{\boldsymbol{x}} &:\equiv \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|\Diamond A|_{\boldsymbol{y}} &:\equiv \exists \boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}
\end{aligned}$$

As an immediate consequence,

$$|(A \to_k B) :\equiv (\Box A \to B)|^{\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{v}} \quad :\equiv \quad \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \to |B|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{v}}$$

The following, although part of this definition, are only of secondary interest in this paper, which is focused on the use of modal operators:

$$|\forall_+ z\, A(z)|^{\boldsymbol{f}}_{\boldsymbol{y}} :\equiv \forall z\, |A(z)|^{\boldsymbol{f}z}_{\boldsymbol{y}} \qquad |\forall_- z\, A(z)|^{\boldsymbol{x}}_{z,\boldsymbol{y}} :\equiv |A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

$$|\forall_\emptyset z\, A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}} :\equiv \forall z\, |A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}} \qquad |\exists_\emptyset z\, A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}} :\equiv \exists z\, |A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

As an issue of style, we promote the use of modal operators whenever possible instead of the above partly (or non) computational quantifiers $\forall_+$, $\forall_-$, $\forall_\emptyset$ and $\exists_\emptyset$. Note that the usual restriction on the contraction rule, that the interpretation of the contraction formula must be quantifier-free, now has to take into account also the interpretation of the modal operators, as this introduces new quantifiers. Thus, refutation relevant contraction formulas cannot include modal operators, see also [6].

Program extraction by modal Dialectica is described by the following theorem, which is an immediate adaptation of the soundness theorem for light Dialectica from [6].

Theorem 1 (Soundness of modal Dialectica interpretation).
Let $A_0, A_1, \dots, A_n$ be a sequence of formulas in $\mathcal{F}_l$ with $\boldsymbol{w}$ all their free variables. If the sequent $a_1 : A_1, \dots, a_n : A_n \vdash_l A_0$ is provable in MA_l, then terms $\boldsymbol{t}_0, \dots, \boldsymbol{t}_n$ can be automatically synthesized from its formal proof, such that the translated sequent $a_1 : |A_1|^{\boldsymbol{x}_1}_{\boldsymbol{t}_1}, \dots, a_n : |A_n|^{\boldsymbol{x}_n}_{\boldsymbol{t}_n} \vdash |A_0|^{\boldsymbol{t}_0}_{\boldsymbol{x}_0}$ is provable in MA, where the following free variable condition (c) *holds:* $\mathsf{FV}(\boldsymbol{t}_i) \subseteq \{\boldsymbol{w}, \boldsymbol{x}_0, \dots, \boldsymbol{x}_n\}$ *and* $\boldsymbol{x}_0 \notin \mathsf{FV}(\boldsymbol{t}_0)$*. Here $\boldsymbol{x}_0, \dots, \boldsymbol{x}_n$ are tuples of fresh variables, s.t. equal avars share a common such tuple.*

We can now complete the definition of $\Box^i$: the restriction is that $\boldsymbol{x}_0 \notin \cup_{i=1}^n \mathsf{FV}(\boldsymbol{t}_i)$ in the translated premise sequent $a_1 : |A_1|^{\boldsymbol{x}_1}_{\boldsymbol{t}_1}, \dots, a_n : |A_n|^{\boldsymbol{x}_n}_{\boldsymbol{t}_n} \vdash |A_0|^{\boldsymbol{t}_0}_{\boldsymbol{x}_0}$. This ensures that the introduction rule $\forall^i$ can be applied for variables $\boldsymbol{x}_0$ and thus the conclusion sequent $a_1 : A_1, \dots, a_n : A_n \vdash_l \Box A_0$ is witnessed by the same realizers as the premise.

Theorem 2. *Axioms* `AxT`*,* `AxT'`*,* `Ax4`*,* `Ax4'` *and* `AxK` *are realizable under the above translation.*

Proof: The translation of `AxT` is $|\Box A \to A|^{\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{y}} \equiv \forall \boldsymbol{v} |A|^{\boldsymbol{x}}_{\boldsymbol{v}} \to |A|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{y}}$ and we can take $\boldsymbol{g}$ to be the identity $\lambda \boldsymbol{x}.\, \boldsymbol{x}$. Similarly, the translation of `AxT'` is $|A \to \Diamond A|^{\boldsymbol{f}}_{\boldsymbol{x},\boldsymbol{y}} \equiv |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}\boldsymbol{y}} \to \exists \boldsymbol{u} |A|^{\boldsymbol{u}}_{\boldsymbol{y}}$ and we can take $\boldsymbol{f}$ to be the projection $\lambda \boldsymbol{x}\boldsymbol{y}.\, \boldsymbol{y}$. For `Ax4` and `Ax4'` it is immediate that $|\Box A| \equiv |\Box\Box A|$ and thus the realizer is again the identity. In the translation of `AxK` below, we take $\boldsymbol{U} :\equiv \lambda \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{x}.\, \boldsymbol{g}\boldsymbol{x}$, which can easily be proved to be a realizer.

$$
\begin{aligned}
|\mathtt{AxK}| \;&\equiv\; [\forall \boldsymbol{x}, \boldsymbol{v}(|A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}\boldsymbol{v}} \to |B|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{v}}) \wedge \forall \boldsymbol{y} |A|^{\boldsymbol{x}'}_{\boldsymbol{y}}\,]^{\boldsymbol{f},\boldsymbol{g},\boldsymbol{x}'} \to \forall \boldsymbol{v}' |B|^{\boldsymbol{u}}_{\boldsymbol{v}'} \quad \equiv \\
&\equiv\; [\forall \boldsymbol{x}, \boldsymbol{v}(|A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}\boldsymbol{v}} \to |B|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{v}}) \wedge \forall \boldsymbol{y} |A|^{\boldsymbol{x}'}_{\boldsymbol{y}} \to \forall \boldsymbol{v}' |B|^{\boldsymbol{U}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{x}')}_{\boldsymbol{v}'}\,]^{\boldsymbol{U}}_{\boldsymbol{f},\boldsymbol{g},\boldsymbol{x}'}
\end{aligned}
$$

Theorem 3. *Axioms* `Ax5` : $\Diamond A \to \Box\Diamond A$ *is generally not realizable under the modal Dialectica translation.*

Proof: The translation of `Ax5` is a formula of shape $B(\boldsymbol{x}) \to \forall \boldsymbol{y} B(\boldsymbol{y})$ which only holds true when $\boldsymbol{x}$ is the empty tuple, special case when `Ax5` requires no realizer at all.

Notice that in systems MA_l, MA the possibility operator $\Diamond$ cannot be defined in terms of the necessity operator $\Box$ via double negation. Only in the classical variants of MA_l and MA can we see $\Diamond$ as $\neg\Box\neg$, which would be part of a double negation translation for MA_l, see also Section 3 below. Also notice that $\Diamond\exists \boldsymbol{x} A$ is similar to Berger's uniform existence $\{\exists \boldsymbol{x}\} A$ from [1], where one does not care about the witness for $\exists \boldsymbol{x}$ (which is actually deleted from the extraction). We can thus see $\Diamond$ as an extension of Berger's tool to more general formulas than just existential ones. On the other hand there are situations when $\Box$ and $\Diamond$ are too general tools and separate annotations for each quantifier are a better answer for the problem at hand. In some of these cases it may still be possible to use the modal operators if one changes the input specification and its proof.

2 Examples

The weak extensionality/compatibility can be expressed in MA_l as the following axiom: $\mathtt{CmpAx}: \ \Box(\boldsymbol{x} =_\rho \boldsymbol{y}) \ \rightarrow \ A(\boldsymbol{x}) \rightarrow A(\boldsymbol{y})$. It is easy to see that CmpAx is realizable under modal Dialectica by identity functional(s).

As first argued in [5], induction for natural numbers should always be treated in a Modified Realizability style, hence the induction rule $\mathtt{Ind}_\iota$ of MA_l should rather be defined with "Kreisel" implications at the step, as follows (the same restriction as in [6] applies on the rule below - $|\Delta|$ must contain only quantifier-free formulas):

$$\frac{\Gamma \vdash A(\mathtt{0}) \quad \Delta \vdash \Box A(n) \rightarrow A(\mathtt{S}n)}{\Gamma, \Delta \vdash A(n)} \ \mathtt{Ind}_\iota$$

The treatment of $\mathtt{Ind}_\iota$ under modal Dialectica, is somewhat easier than in [6]: the proof is just the same, but the realizing terms are much simpler, as the unnecessary realizers are no longer included. Thus the optimization here is by elimination of redundancy.

In [5] the following class of examples was considered: theorems of the form

$$\forall \boldsymbol{x} A \rightarrow \forall \boldsymbol{y} B \rightarrow \forall \boldsymbol{z} C \tag{1}$$

possibly with parameters, where the negative information on $\boldsymbol{x}$ is irrelevant, while the one on $\boldsymbol{y}$ is of our interest. Then it must be possible to adapt the proof of (1) to a proof in MA_l of $(\Box \forall \boldsymbol{x} A) \rightarrow \forall \boldsymbol{y} B \rightarrow \forall \boldsymbol{z} C$. As noticed by Oliva in [9], the Fibonacci example first treated with Dialectica in [4] falls into this category.

3 Remarks

Modal Dialectica provides the means of using both Modified Realizability and Gödel's Dialectica at the same time for more efficient program extraction. This was already the case for the hybrid Dialectica of [5], but here we eliminate the detour to the linear logic sublevel. Disregarding the partially or non-computational quantifiers, modal Dialectica represents (directly at the intuitionistic logic level) the right combination of the original proof interpretations, with the possibility of carrying out both in a sound way. All one needs is that some implications can be seen as Kreisel implications $\rightarrow_k$.

System MA_l is intuitionistic, as stability $\neg\neg A \rightarrow A$ cannot be proved for formulas with strong existentials. The classical counterpart of MA_l would include stability as axiom and then it could be interpreted by modal Dialectica via some negative translation, e.g., replacing strong existentials $\exists$ with weak existentials $\neg\forall\neg$, see [3].

The usual restriction on the introduction rule for the necessity operator is that $\Gamma \equiv \emptyset$. In the natural deduction presentation of modal logic, $\Box^i$ cannot be unrestricted or

$A \to \Box A$ becomes a theorem, which cannot be. Our restriction on $\Box^i$ seems to be weaker, as virtually allows any Γ, but the variables condition also needs to be taken into account. Moreover, our restriction on contraction is not present in the usual first-order modal logic systems. See [7] for extensive comments on the design of formalisms for modal logic, particularly on the yet-unsatisfactory definition of necessity introduction in Natural Deduction systems. Modulo contraction, we give the optimal restriction for $\Box^i$, but this does not solve the issue for general, fully-fledged first-order modal logics.

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