Classical models are particular cases of Probabilistic models

Dan Hernest
Research Institute for Informatics
Bd.Maresal Averescu Nr.8-10, 71316 Bucharest
E-mail: danher@phobos.cs.unibuc.ro

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Abstract

Probabilistic model theory is an extension of Model theory for first order predicate calculus. The basic idea is to replace truth valuations with probabilities defined on sets of sentences. For propositional calculus this reduces to defining a probability on the Lindenbaum-Tarski algebra associated with the system, but for predicate calculus one needs to express the way such a probability behaves for sentences with quantifiers. Haim Gaifman introduced such a condition and sketched the way that probabilistic models could be regarded as extensions of classical models. This paper is concerned with clarifying Gaifman results. We introduce the notion of "boolean theory" for the classical case, that would be used in defining the "probabilistic theories", such that classical theories are particular cases of probabilistic theories. Also probabilistic models are defined such a way that classical models are particular cases of them, namely by considering two valued probabilities (with only the values 0 and 1).

1 Introduction

In this paper we will consider a given countable set of variables $\mathcal{V} = \{v_0, v_1, \ldots\}$ and a given set of relational symbols $\mathcal{R}$, the same for all first order predicate languages (FOPL) that might appear in the sequel. We will denote these FOPL with the same symbol we use to denote their set of constant symbols. We will also consider a given set of constant symbols, denoted by $\mathcal{C}$. If no otherway specified, the implicit FOPL shall be $\mathcal{C}$.

We will use in the sequel the following notations. If $\mathbf{A}$ is a Boolean algebra and $M \subseteq A$, then $\overline{M}$ stands for the Boolean subalgebra of A generated by M. Let M be a set of constant symbols. We will denote by $\mathbf{E}(M)$ the set af all sentences of M, by $\mathbf{E_0}(M)$ the set of quantifier free sentences of M , by $\mathbf{At}(M)$ the set of all atomic formulas of M , that is formulas of the kind $t_1 = t_2$ or $R(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are variables or constant symbols and $R \in \mathcal{R}$ is an $n-$ ary relation symbol , by $\mathbf{At_0}(M)$ the set of atomic sentences of M , by $\mathbf{LTF}(M)$ the Lindenbaum-Tarski algebra of the language M, that is $\mathbf{F}(M)/\sim_M$, where $\phi \sim_M \psi$ iff $\vdash (\phi \leftrightarrow \psi)$. $[\phi]$ stands for the equivalence class of the formula ϕ with respect to the relation

$\sim_M$, $[\mathcal{H}] := \{ [\phi] \,|\, \phi \in \mathcal{H} \}$, where $\mathcal{H}$ is a set of formulas, $\mathbf{LT}(M)$ is the Boolean subalgebra of $\mathbf{LTF}(M)$ who's elements are all the equivalence classes of the sentences (it is isomorphic to $\mathbf{E}(M)/\sim_M$), $\mathbf{LT_0}(M) := \{ [\sigma] \mid [\sigma] \in \mathbf{LT}(M)$ and σ is quantifier-free $\}$. The letters x, y, eventually naturally indexed, stand for variables from the set $\mathcal{V}$. In the case we will use a sequence of variables $x_1, \ldots, x_k$ we will suppose the elements are mutually distinct.

We have that $\mathbf{LT_0}(M) = \overline{[\mathbf{E_0}(M)]} \simeq \mathbf{E_0}(M)/\sim_M$, $\mathbf{LT_0}(M)$ is a Boolean subalgebra of $\mathbf{LT}(M)$, $\mathbf{LT_0}(M) = \overline{[\mathbf{At_0}(M)]}$. If $M' \subseteq M$ then $\mathbf{LT}(M') \leq \mathbf{LT}(M)$ and $\mathbf{LT_0}(M') \leq \mathbf{LT_0}(M)$.

We also remind that if $\mathbf{B}$ is a Boolean algebra and $\mu : B \mapsto [0,1]$ then μ is a *probability* on $\mathbf{B}$ if $\mu(b_1 \vee b_2) = \mu(b_1) + \mu(b_2)$ for any $b_1, b_2 \in B,\ b_1 \wedge b_2 = 0$ and $\mu(1) = 1$. We say that a probability $\mu : B \mapsto [0,1]$ is binary if the image of μ is the set $\{0,1\}$.

Theorem 1.1 Let us provide the set $\{0,1\}$ with the usual Boolean algebra structure and denote it by L_2. A mapping $\mu : B \mapsto L_2$ is a binary probability iff it is a morphism of Boolean algebras.

2 Classical models and theories

This chapter is a completion of the results obtained in Chapter 3 from [He98], especially regarding Boolean sets of sentences and Boolean theories. We shortly remind the main definitions and results obtained there.

We denote by $\mathbf{SyC}(\Sigma)$ the set of all syntactical consequences of a set Σ of sentences. We say that Σ is *inconsistent* if $\mathbf{SyC}(\Sigma) = \mathbf{E}$. We say that Σ is *consistent* if it is not inconsistent. We have that Σ is consistent iff for any sentence $\phi \in \mathbf{E}$ at most one of $\phi \in \mathbf{SyC}(\Sigma)$, $(\neg\phi) \in \mathbf{SyC}(\Sigma)$ is true. A *theory* of the language M is a consistent set of sentences of M. Let us denote by $\mathbf{Th}(M)$ the set of all the theories of the language M. Let $Sup(B) := \bigcup_{[\phi]\in B}[\phi]$ be the *support* of the set $B \subseteq \mathbf{LT}(M)$. A set Σ of sentences is said to be *Boolean* if it is the support of some Boolean subalgebra of $\mathbf{LT}$. We denote by $\mathbf{BSS}(M)$ the set of all Boolean sets of sentences of M.

If $\Sigma \in \mathbf{BSS}$ then for any $\phi, \psi \in \Sigma$ it is true that the sentences $(\neg\phi)$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \to \psi)$ and $(\phi \leftrightarrow \psi)$ belong to Σ. If $\phi \in \Sigma$ and $\psi \in \mathbf{E}$ so that $\vdash (\phi \leftrightarrow \psi)$ then $\psi \in \Sigma$.

The intersection of a family of BSS of is a BSS too. We say that the set $\mathbf{BlC}(\Sigma) := \bigcap_{\Sigma\subseteq\mathcal{S}\in\mathbf{BSS}} \mathcal{S}$ is the *Boolean closure* of a set Σ of sentences. It is the least BSS that containes Σ. The mapping $\mathbf{BlC} : \mathcal{P}(\mathbf{E}) \mapsto \mathcal{P}(\mathbf{E})$ is a closure operator. It is true that $\mathbf{E_0}(M) = \mathbf{BlC}(\mathbf{At_0}(M))$. We also have that $\mathbf{BlC}(\Sigma) = Sup(\overline{[\Sigma]}) = Sup(\{ \vee_{i=1}^{n}(\wedge_{j=1}^{m_i}[\sigma_{i,j}]) \,|\, \sigma_{i,j} \in \Sigma$ or $\sigma_{i,j} \in \neg\Sigma \})$. We say that a theory $\mathcal{T}$ is a *Boolean theory* if $\mathcal{T} \cup \{ \phi \in \mathbf{E} \,|\, (\neg\phi) \in \mathcal{T} \} \in \mathbf{BSS}$.

Let us denote by $\mathbf{BlT}(M)$ the set of all Boolean theories of the language M. A theory $\mathcal{T}$ is a Boolean theory iff $\mathbf{BlC}(\mathcal{T}) = \mathcal{T} \cup \{ \phi \in \mathbf{E} \,|\, (\neg\phi) \in \mathcal{T} \}$. If $\mathcal{T} \in \mathbf{BlT}$, then $\neg\mathcal{T} \subseteq \{ \phi \in \mathbf{E} \,|\, (\neg\phi) \in \mathcal{T} \}$.

Definition 2.1 Let us denote by $\mathbf{BiP}(M)$ the set of all binary probabilities defined on Boolean subalgebras of $\mathbf{LT}(M)$.

Theorem 2.2 1) Let $b : \mathbf{Th} \mapsto \mathbf{BiP}$ a mapping defined by $b(\mathcal{T}) := h$, where the domain of h is $B := \overline{[\mathcal{T}]} = \{ \vee_{i=1}^{n}(\wedge_{j=1}^{m_i}[\phi_{i,j}]) \,|\, \phi_{i,j} \in \mathcal{T}$ or $\phi_{i,j} \in \neg\mathcal{T} \}$ and h is defined by $h(\vee_{i=1}^{n}(\wedge_{j=1}^{m_i}[\phi_{i,j}])) :=$

$\vee_{i=1}^{n}(\wedge_{j=1}^{m_i}h([\phi_{i,j}]))$ where $h([\phi_{i,j}]) := \begin{cases} 1, \text{ if } \phi_{i,j} \in \mathcal{T} \\ 0, \text{ if } \phi_{i,j} \in \neg\mathcal{T} \end{cases}$. We have that $[\mathcal{T}] \subseteq B$, h is well defined, $h([\phi]) = 1$ for any $\phi \in \mathcal{T}$ and h is a binary probability (b is well-defined).
2) Let $t : \mathbf{BiP} \mapsto \mathbf{BlT}$ defined by $t(h) := Sup(h^{-1}(1))$. We have that t is well-defined, that is $t(h)$ is a Boolean theory.
3) The mappings b and t define a one-one correspondence between **BiP** and **BlT**.

We present in the sequel some definitions and results that complete the ones exposed above. The following Lemma offers a generative mechanism for $\mathbf{BlC}(\Sigma)$, as the closure of Σ to some set of rules.

Lemma 2.3 Let Σ be a set of sentences. Then $\mathbf{BlC}(\Sigma)$ is the least set of sentences $\mathcal{H}$ that respects the following rules:
1) $\Sigma \subseteq \mathcal{H}$.
2) If $\phi, \psi \in \mathcal{H}$ then $(\neg\phi) \in \mathcal{H}$ and $(\phi \rightarrow \psi) \in \mathcal{H}$.
3) If $\phi \in \mathcal{H}$ and $\psi \in \mathbf{E}(M)$ so that $\vdash (\phi \leftrightarrow \psi)$ then $\psi \in \mathcal{H}$.

Proof: Indeed, $\mathbf{BlC}(\Sigma)$ respects the rule 1) by definition and the rules 2) and 3) by the fact that it is a boolean set of sentences. Supppose $\mathcal{H}$ is a set of sentences that respects 1),2),3). We prove that $\mathbf{BlC}(\Sigma) \subseteq \mathcal{H}$. Indeed, from rules 2) and 3) we deduce that $\mathcal{H}$ is a Boolean set of sentences , in fact $[\mathcal{H}]$ is a Boolean subalgebra (rule 2)) of **LT**, and $\mathcal{H} = Sup([\mathcal{H}])$ (rule 3)).

Remark 2.4 We coluld define theories of a language M starting from the binary probabilities defined on Boolean subalgebras of $\mathbf{LT}(M)$:
1) A set of sentences is a Boolean theory iff it is the image of some element of **BiP** through the mapping t.
2) A set of sentences is a theory iff it is a subset of the image of some element of **BiP** through the mapping t.

The Remark above allows us to generalize the notion of *theory* to the notion of *probabilistic theory*. We remind that a (classical)*model* of the language M is an ordered pair $< A, \mathcal{I} >$ where A is a set and $\mathcal{I}$ is a *interpretation mapping* with domain $\mathcal{C} \cup \mathcal{R}$ that associates to $c \in M$ an element $\mathcal{I}(c) \in A$ and to $R \in \mathcal{R}$ an n-ary relation I(R) on A.

3 Probabilistic models and theories

Definition 3.1 By a *Boolean probabilistic theory* we will understand a probability $\mu : B \mapsto [0, 1]$ where $B \leq \mathbf{LT}$ is a Boolean subalgebra of **LT**. We will say that B is the *domain* of μ and denote it by $B = D\mu$. We also denote by $\mathbf{BlPT}(M)$ the set of all Boolean probabilistic theories of M.

The notion of **BlPT** is a generalization of the notion of **BlT**. This is because, according to 2.2, there is a one-one corespondence between **BlT** and binary **BlPT**.

Definition 3.2 1) By a *probabilistic theory* T we understand the restriction of a **BlPT** μ_T to a subset of its domain. That is a set of ordered pairs $< [\phi], \mu([\phi]) >$, where $[\phi] \in \mathbf{LT}$ and $\mu([\phi]) \in [0,1]$ such that there is an extension $\nu : [\pi_1(T)]_{BL} \mapsto [0,1]$ of μ which is also a probability. Let us denote by $\mathbf{PT}(M)$ the set of all the probabilistic theories of M.
2) In the case μ_T is binary, we say that T is a *binary probabilistic theory*. Denote by $\mathbf{BiPT}(M)$ the set of all the binary probabilistic theories of M.

Theorem 3.3 Define the relation $\approx$ on the set **BiPT** by $T \approx T'$ iff for any ordered pair $< [\phi], \alpha > \in \mathbf{LT} \times \{0,1\}$, if $< [\phi], \alpha > \in T$, then $< [\phi], \alpha > \in T'$ or $< \neg[\phi], 1-\alpha > \in T'$ and conversely, if $< [\phi], \alpha > \in T'$, then $< [\phi], \alpha > \in T$ or $< \neg[\phi], 1-\alpha > \in T$. Define the relation $\sim$ on **Th** by $\mathcal{T} \sim \mathcal{T}'$ iff $[\mathcal{T}] = [\mathcal{T}']$. We have that $\approx$ and $\sim$ are equivalence relations and there is a one-one correspondence between $\mathbf{Th}/\sim$ and $\mathbf{BiPT}/\approx$.

Proof: It is easy to verify that both $\approx$ and $\sim$ are equivalence relations. Let $\mathcal{T}$ be a theory of M. Observe from Theorem 2.2 that $b(\mathcal{T})$ is a binary probability and that $[\mathcal{T}] \subseteq b(\mathcal{T})^{-1}(1)$. Define $p : \mathbf{Th}/\sim \mapsto \mathbf{BiPT}/\approx$ by $p(\mathcal{T}/\sim) := (b(\mathcal{T})/[\mathcal{T}])/\approx$. Observe that p is well defined. Conversely, if T is a **BiPT**, construct first the **BiPT** T' such that $T \approx T'$ and $\pi_2(T') = \{1\}$. Define $q : \mathbf{BiPT}/\approx \mapsto \mathbf{Th}/\sim$ by $q(T/\approx) := Sup(\pi_1(T'))/\sim$. Observe that q is well defined. We proove now that p and q establish a bijection. Since $[\mathcal{T}] = [Sup([\mathcal{T}])]$ obtain that $q \circ p = 1$. Also $p(q(T/\approx)) = T'/\approx$, wherefrom $p \circ q = 1$.

Definition 3.4 Let M be a set and $\mu : \mathbf{LT}(M) \mapsto [0,1]$ a probability. We say that μ satisfies the *Gaifman condition* iff for any $\phi(x) \in \mathbf{F}(M)$ such that x is its only free variable, it is true that $\mu([(\exists x)\phi(x)]) = \sup \{ \mu([\vee_{j=1}^{n} \phi(a_j)]) \mid a_1, \ldots, a_n \in M \}$. We will say that μ is a Gaifman probability, or shortly, a G-probability.

Definition 3.5 1) We define a *probabilistic model* of $\mathcal{C}$, shortly P-model, as an ordered triple $\mathcal{A} =< A, \mathcal{J}, \mu >$ where $\mathcal{J} : \mathcal{C} \mapsto A$ and $\mu : \mathbf{LT}(A) \mapsto [0,1]$ is a G-probability.
2) We define a *strict equality model*, shortly SE-model as a P-model $< A, \mathcal{J}, \mu >$ such that $\mu([a_1 = a_2]) = 0$ for any $a_1, a_2 \in A$, $a_1 \neq a_2$.
3) We define a *Gaifman model*, shortly G-model as a P-model $< U, \mathcal{I}, \nu >$ such that $\mathcal{C} \subseteq U$ and $\mathcal{I} : \mathcal{C} \mapsto U$ is the canonical injection $\mathcal{I}(c) = c$ for any $c \in \mathcal{C}$.

We can denote a G-model by $< U, \nu >$, where $\mathcal{C} \subseteq U$ and $\nu : \mathbf{LT}(U) \mapsto [0,1]$ is a G-probability. The following result is prooved in [Ga62].

Theorem 3.6 (Gaifman extension theorem). Let M be a set and let $m : \mathbf{LT_0}(M) \mapsto [0,1]$ be a probability. There exists a unique G-probability $m^* : \mathbf{LT}(M) \mapsto [0,1]$ that extends m to $\mathbf{LT}(M)$

Theorem 3.7 1) Let $\mathcal{A} =< A, \mathcal{I} >$ be a (classical) model. Define the mapping $\mu : \mathbf{LT}(A) \mapsto \{0,1\}$ by $\mu([\phi(a_1, \ldots, a_n)]) := 1$ iff $\mathcal{A} \models \phi[a_1, \ldots, a_n]$. Then μ is a binary probability and $< A, \mathcal{I}/\mathcal{C}, \mu >$ is a SE-model.
2) Let $\mathcal{A} =< A, \mathcal{J}, \mu >$ be a SE-model such that μ is a binary probability. Define the interpretation $\mathcal{I} : \mathcal{C} \cup \mathcal{R}$ by $\mathcal{I}(c) := \mathcal{J}(c)$ for any $c \in \mathcal{C}$ and $\mathcal{I}(R)(a_1, \ldots, a_n)$ iff $\mu([R(a_1, \ldots, a_n)]) = 1$. We have just obtained the (classical) model $< A, \mathcal{I} >$.

3) The constructions from 1) and 2) above establish a bijection between the class of classical models and the class of SE-models of $\mathcal{C}$.
4) Denote by $\mathcal{A}$ both the classical model and the corresponding SE-model. We then have for any $\phi \in \mathbf{E}$ that $\mathcal{A} \models \phi(c_1, \ldots, c_n)$ iff $\mu_A([\phi(\mathcal{I}_A(c_1), \ldots, \mathcal{I}_A(c_n))]) = 1$.
5) In the case of a SE-model, denote it by $\mathcal{A}$, the Gaifman condition becomes $\mathcal{A} \models (\exists x)\phi(x)$ iff there exists $a \in A$ such that $\mathcal{A} \models \phi[a]$.

Proof:
1) The application μ is well defined because if $[\phi] = [\psi]$ then $\models (\phi \leftrightarrow \psi)$, therefore $\mathcal{A} \models \phi$ iff $\mathcal{A} \models \psi$. The fact that μ is a probability comes from the definition of satisfaction. The Gaifman condition is satisfied because: $\mu([(\exists x)\phi(x)]) = 1$ iff $\mathcal{A} \models (\exists x)\phi(x)$ iff there exists $a \in A$ such that $\mathcal{A} \models \phi[a]$, iff there exists $a \in A$ such that $\mu([\phi(a)]) = 1$, iff there exists $a_1, \ldots, a_n \in A$ such that $\mu([\vee_{j=1}^n \phi(a_j)]) = 1$, iff $\sup \{ \mu([\vee_{j=1}^n \phi(a_j)]) \mid a_1, \ldots, a_n \in A \} = 1$. It is clear that $\mu([a_1 = a_2]) = 1$ iff $a_1 = a_2$.
3) Denote by p the application that associates a SE-model to a model and by q the application that associates a model to a SE-model. It is clear that $q \circ p = 1$, because $\mathcal{A} \models R[a_1, \ldots, a_n]$ iff $I_{\mathcal{A}}(R)(a_1, \ldots, a_n)$. The fact that $p \circ q = 1$ could be proved with a similar argument , using Theorem 3.6 and the fact that for binary probabilities it is sufficient that two probabilities are equal on $[\mathbf{At_0}(A)]$ for them to be equal on $\mathbf{LT_0}(A)$.
4) $\mathcal{A} \models \phi(c_1, \ldots, c_n)$ iff $\mathcal{A} \models \phi[\mathcal{I}_A(c_1), \ldots, \mathcal{I}_A(c_n)]$ iff $\mu_A([\phi(\mathcal{I}_A(c_1), \ldots, \mathcal{I}_A(c_n))]) = 1$.

The notion of *probabilistic model* is a generalization of the classical notion of model. The defining properties of probability, namely $\mu(a \vee b) = \mu(a) + \mu(b)$ for $a \wedge b = 0$ and $\mu(1) = 1$ generalize the definition of satisfaction for sentences $\phi \vee \psi$, respectively $(\neg\phi)$, and the Gaifman condition generalizes the definition of satisfaction for sentences $(\exists x)\phi(x)$.

We will introduce in the sequel the notion of *satisfaction*, in its probabilistic version.

Definition 3.8 Let $T : D(T) \mapsto [0,1]$ be a **PT** and let $\mathcal{A} =< A, \mathcal{I}, \mu >$ be a P-model. If $T([\phi(c_1, \ldots, c_n)]) = \mu([\phi(\mathcal{I}(c_1), \ldots, \mathcal{I}(c_n)])$ for any sentence $\phi \in D(T)$, we say that $\mathcal{A}$ satisfies T or, equivalently, that $\mathcal{A}$ is a model of T. We denote it by $\mathcal{A} \models T$.

If $\mathcal{A} =< U, \nu >$ is a G-model, then $\mathcal{A} \models T$ iff $T = \nu / D(T)$.

Theorem 3.9 (Gaifman completeness theorem). Every $T \in \mathbf{PT}$ has a P-model.

Proof: Let μ_T be the **BlPT** associated with T. In [Ga62] is proved a theorem that states that μ_T admits an extension to a G-probability $\nu : U \mapsto [0,1]$, such that $\mathcal{C} \subseteq U$. Then $< U, \nu >$ is a G-model and $\nu / D(T) = \mu_T / D(T) = T$.

The G-model $< U, \nu >$ constructed above has the property that for any $\phi(x) \in \mathbf{F}(U)$ having x as its only free variable, there exists $a \in U - \mathcal{C}$ such that $\nu([(\exists x)\psi(x)]) = \nu([\psi(a)])$, iff for any $\phi(x_1, \ldots, x_k) \in \mathbf{F}(U)$ having $x_1, \ldots, x_k$ as its only free variables there exists $a_1, \ldots, a_k \in U - \mathcal{C}$ such that $\nu([(\exists x_1 \ldots x_k)\psi(x_1, \ldots, x_k)]) = \nu([\psi(a_1, \ldots, a_k)])$.

We have extended the notion of *probabilistic model* in the Gaifman's terminology from [Ga62] but we kept its sense and denominated here as *Gaifman model*. This extension was needed for the establishment of the one-one correspondence between classical models and strict equality models. The generalization of the notion of *model* from its classical sense to the probabilistic sense comes more natural this way. We proove in the sequel that our notion of probabilistic model is someway equivalent to Gaifman's.

Definition 3.10 We say that two probabilistic models $\mathcal{A} =< A, \mathcal{I}_A, \mu_A >$ and $\mathcal{B} =< B, \mathcal{I}_B, \mu_B >$ are **elementary equivalent** and write it $\mathcal{A} \equiv \mathcal{B}$ iff $\mu_A([\phi(\mathcal{I}_A(c_1), \ldots, \mathcal{I}_A(c_n))]) = \mu_B([\phi(\mathcal{I}_B(c_1), \ldots, \mathcal{I}_B(c_n))])$ for any sentence $\phi(c_1, \ldots, c_n) \in \mathbf{E}$.

The notion of probabilistic models' *elementary equivalence* is the generalization of the notion of *elementary equivalence* from classical Model Theory. That is, two classical models are classicaly elementary equivalent iff their strict equality corespondents are probabilistically elementary equivalent.

Theorem 3.11 Any probabilistic model is elementary equivalent to a G-model.

Proof: Let $\mathcal{A} =< A, \mathcal{I}, \mu >$ be a probabilistic model. Suppose that $\mathcal{C}$ and A are disjoint sets. Let $U := \mathcal{C} \cup (A - \mathcal{I}(\mathcal{C}))$ and $\nu : \mathbf{LT}(U) \mapsto [0,1]$ defined by $\nu([\phi(u_1, \ldots, u_n]) := \mu([\phi(\mathcal{I}(u_1), \ldots, \mathcal{I}(u_n))])$, where he have extended $\mathcal{I}$ from $\mathcal{C}$ to U by defining $\mathcal{I}(u) := u$ for any $u \in A - \mathcal{I}(\mathcal{C})$. Then $< U, \nu >$ is a Gaifman model,elementary equivalent with $\mathcal{A}$. First, the mapping ν is well defined and it is a probability. Indeed, if $[\phi(u_1, \ldots, u_n)] = 1$ then $\vdash \phi(u_1, \ldots, u_n)$ iff $\vdash (\forall x_1 \ldots x_n)\phi(x_1, \ldots, x_n)$, which entailes $\vdash \phi(\mathcal{I}(u_1), \ldots, \mathcal{I}(u_n))$, therefore $[\phi(\mathcal{I}(u_1), \ldots, \mathcal{I}(u_n))] = 1$. Using that $[\phi] = 0$ iff $[(\neg\phi)] = 1$ we obtain that if $[\phi(u_1, \ldots, u_n)] = 0$ then $[\phi(\mathcal{I}(u_1), \ldots, \mathcal{I}(u_n))] = 0$.

The well-defining of ν comes from the fact that $[\phi(u_1, \ldots, u_n)] = [\psi(v_1, \ldots, v_m)]$ iff $[\phi(u_1, \ldots, u_n)] \leftrightarrow [\psi(v_1, \ldots, v_m)] = 1$ iff $[\phi(u_1, \ldots, u_n) \leftrightarrow \psi(v_1, \ldots, v_m)] = 1$. The last identity implies $[\phi(\mathcal{I}(u_1), \ldots, \mathcal{I}(u_n)) \leftrightarrow \psi(\mathcal{I}(v_1), \ldots, \mathcal{I}(v_m))] = 1$ iff $[\phi(\mathcal{I}(u_1), \ldots, \mathcal{I}(u_n))] = = [\psi(\mathcal{I}(v_1), \ldots, \mathcal{I}(v_m))]$. Let $\phi, \psi \in \mathbf{E}(U)$ such that $[\phi] \wedge [\psi] = 0$, iff $[\phi \wedge \psi] = 0$, therefore $[\phi(\mathcal{I}(u_1), \ldots, \mathcal{I}(u_n)) \wedge \psi(\mathcal{I}(v_1), \ldots, \mathcal{I}(v_m))] = 0$ iff $[\phi(\mathcal{I}(u_1), \ldots, \mathcal{I}(u_n))] \wedge [\psi(\mathcal{I}(v_1), \ldots, \mathcal{I}(v_m))] = 0$. Denoting $\mathcal{I}(\phi(u_1, \ldots, u_n)) := \phi(\mathcal{I}(u_1), \ldots, \mathcal{I}(u_n))$ we obtain that $[\mathcal{I}(\phi)] \wedge [\mathcal{I}(\psi)] = 0$. Then $\nu([\phi] \vee [\psi]) = \nu([\phi \vee \psi]) = \mu([\mathcal{I}(\phi \vee \psi)] = \mu([\mathcal{I}(\phi) \vee \mathcal{I}(\psi)] = \mu([\mathcal{I}(\phi)] \vee [\mathcal{I}(\psi)] = \mu([\mathcal{I}(\phi)]) + \mu([\mathcal{I}(\psi)]) = \nu([\phi]) + \nu([\psi])$. We proove that ν is a G-probability: $\nu([(\exists x)\phi(x)]) = \mu([(\exists x)\phi(x)]) = \sup \{ \mu([\vee_{j=1}^n \phi(a_j)]) \mid a_1, \ldots, a_n \in A \}$. The function $\mathcal{I} : U \mapsto A$ is surjective, therefore, for all $a_1, \ldots, a_n \in A$ there exist $u_1, \ldots, u_n \in U$ such that $a_1 = \mathcal{I}(u_1), \ldots, a_n = \mathcal{I}(u_n)$. We obtain $\nu([(\exists x)\phi(x)]) = \sup \{ \mu([\vee_{j=1}^n \phi(\mathcal{I}(u_j))]) \mid u_1, \ldots, u_n \in U \} = = \sup \{ \nu([\vee_{j=1}^n \phi(u_j)]) \mid u_1, \ldots, u_n \in U \}$. We proove now that the models $\mathcal{A}$ and $< U, \nu >$ are elementary equivalent. Let $\phi(c_1, \ldots, c_n) \in \mathbf{E}(M)$. Then $\nu([\phi(c_1, \ldots, c_n)]) = \mu([\phi(\mathcal{I}(c_1), \ldots, \mathcal{I}(c_n))])$ by definition.

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