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Light monotone Dialectica methods for proof mining

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In view of an enhancement of our implementation on the computer, we explore the possibility of an algorithmic optimization of the various proof-theoretic techniques employed by Kohlenbach for the synthesis of new (and better) effective uniform bounds out of established qualitative proofs in Numerical Functional Analysis. Concretely, we prove that the method (developed by the author in his thesis, as an adaptation to Dialectica interpretations of Berger's original technique for modified realizability and A-translation) of "colouring" some of the quantifiers as "non-computational" extends well to ε-arithmetization, elimination-of-extensionality and model-interpretation.

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1 Introduction

This paper presents the latest results obtained by the author in his recently defended PhD thesis [9]. One of the main achievements of [9] is the development of an optimization for the computer of a proof-theoretic technique successfully[1] employed by Kohlenbach in Numerical and Functional Analysis, named "monotone functional interpretation" (introduced in [14]). The original proof interpretation is due to Gödel [7], under the name "functional interpretation", or even "Dialectica interpretation"[2]. The new "light monotone Dialectica" (LMD) interpretation was implemented in `MinLog` [8] and was successfully used by the author for the extraction of moduli of uniform continuity out of a proof of the hereditarily extensional equality of terms t of Gödel's $\mathbf{T}$ to themselves, see [10].

We here describe how the classical LMD-interpretation could be employed for the more efficient treatment of a larger class of non-trivial ineffective analytical proofs. We follow closely a small part of the exhaustive exposition [20] due to Kohlenbach on the subject of "mining" proofs which use the highly ineffective principle of Weak (also known as "binary") König's Lemma, abbreviated `WKL` (or the related non-standard analytical axioms $\mathtt{F}^-$ and $\mathbf{F}$, of necessary utility in the low-complexity, feasible context). Our goal here is to present the interpretation of proofs which may involve the so-called "non-computational" (abbreviated `nc`) quantifiers. We thus explore the adaptability of the various proof-theoretic techniques employed by Kohlenbach (ε-arithmetization, elimination-of-extensionality, model-interpretation) to the recent `nc` setting. All these techniques adapt fairly straightforwardly to the `nc` context. Yet good care has to be put into the formulation of some peculiar restrictions.

1.1 Formal systems and general notations

We have already mentioned above Gödel's $\mathbf{T}$, from which we use the term system in a variant with λ-abstraction as primitive, closely following [3], but here with monotonic elements added. All terminology and notations below are from Chapter 1 of [9], particularly Sections 1.5 and 1.6.

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[1] See [20, 21, 19] for surveys of these applications of the "monotone Dialectica".

[2] So called after the title of the philosophical journal in which Gödel's work was first published, in German. The monograph [1] is a nice and modern survey in English of a large range of Dialectica's proof-theoretic applications.

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Finite (also known as *simple*) *types* are inductively generated from base types ι for natural numbers and o for booleans by the rule that if σ and τ are types then $(\sigma\tau)$ is a type. Types are denoted by the symbols δ, γ, ρ, σ and τ, which are reserved for such purpose. We make the convention that concatenation is right associative and consequently omit unnecessary parenthesis, writing $\delta\,\sigma\,\tau$ instead of $(\delta(\sigma\tau))$.

Terms $\mathcal{T}_{\mathtt{m}}$ are built from variables and (certain, see Section 1.5 of [9] for details) constants by λ-abstraction and application. We represent the latter as concatenation and we agree that it is left-associative in order to avoid excessive parenthesizing. All variables and constants have an a priori fixed type and terms have a type fixed by their formation. Written term expressions are always assumed to be well-formed in the sense that types match in all applications between sub-terms. We denote terms by r, s, t, S, T, sometime (usually when strictly necessary) with their type indicated in superscript like, e.g., t^τ. We denote by $\mathrm{dg}(t) :\equiv \mathrm{dg}(\tau)$ the *type degree* of t^τ, which is defined for τ by $\mathrm{dg}(\iota) := \mathrm{dg}(o) := 0$ and $\mathrm{dg}(\rho\sigma) := \max\{\mathrm{dg}(\rho)+1\,,\,\mathrm{dg}(\sigma)\}$.

We will tacitly assume the same notations as above also for *tuples* of types, respectively terms (dg now denotes the maximum over the tuple members), since tuples of objects are inherent to our approach to Dialectica, following a tradition that goes back to Gödel. On the other hand, closely following [3, 2], we employ arithmetics for Gödel functionals which are formulated in Natural Deduction, rather than Hilbert-style. *Formulas* are built out of atoms using $\wedge$ (conjunction), $\rightarrow$ (implication), $\forall$ (forall), $\exists$ (strong exists) and for the nc systems also $\forall^{\mathrm{nc}}$ (nc-forall), $\exists^{\mathrm{nc}}$ (nc-exists). *Atomic formulas* are exclusively of shape $\mathtt{at}(t^o)$, with at the unique (and unary) predicate symbol. Upon renaming, we assume that quantified variables occur uniquely in a formula. Relative to occurrences of $\rightarrow$ in a formula A, we distinguish the positive and the negative positions of the quantifiers in A. The positive positions are relative to the right of implications and the negative positions are placed relative to the left of $\rightarrow$. Polarity changes exclusively at the left-hand side of each implication sign of A.

Definition 1.1 A variable of a formula A is *positive* if it is quantified by an $\exists$ or $\exists^{\mathrm{nc}}$ in a positive position or by an $\forall$ or $\forall^{\mathrm{nc}}$ in a negative position. Symmetrically, the variable is called *negative* if it is quantified by an $\exists$ or $\exists^{\mathrm{nc}}$ in a negative position or by an $\forall$ or $\forall^{\mathrm{nc}}$ in a positive position. We denote by $\mathrm{dg}^+(A)$ and $\mathrm{dg}^-(A)$ the maximal type degree of all positive, respectively all negative variables of A.

Definition 1.2 A formula is *pure-nc* if all its quantifiers (if any) are nc. We use an nc in superscript, like A^{nc}, B^{nc} to indicate that a formula is pure-nc.

Definition 1.3 The *regular associate* of a formula A (abbreviated *r-associate*) is obtained by replacing all nc quantifiers of A with their regular-quantifier correspondents. For pure-nc formulas like A^{nc} and B^{nc} the r-associates are usually denoted A, respectively B.

Definition 1.4 By *formula of Delta kind* (abbreviated *Delta formula* or *Delta sentence*) we understand a sentence of shape

$$D :\equiv \forall u^\rho\,(\exists v \leq_\sigma ru)\,\forall w^\tau\; B^{\mathrm{nc}}(u,v,w), \tag{1}$$

where $r^{\rho\sigma}$ is a closed term, $\mathrm{dg}(\tau) \leq 2$ and B^{nc} is a pure-nc formula with all its free variables exactly u, v, w. Let B be the r-associate of B^{nc}. To D we associate the sentences $D_{\mathtt{r}}$ (the r-associate of D), $\overline{D}$ (the choice-strengthening of D, which will be used as intermediate step in the proof of Theorem 2.2) and $\widetilde{D}$ (the r-associate of $\overline{D}$, needed to verify the LMD translation of D) as follows:

$$D_{\mathtt{r}} :\equiv \forall u^\rho\,(\exists v \leq_\sigma ru)\,\forall w^\tau\, B(u,v,w),$$

$$\overline{D} :\equiv (\exists V \leq_{\rho\sigma} r)\,\forall u^\rho\,\forall w^\tau\; B^{\mathrm{nc}}(u, Vu, w),$$

$$\widetilde{D} :\equiv \overline{D}_{\mathtt{r}} :\equiv (\exists V \leq_{\rho\sigma} r)\,\forall u^\rho\,\forall w^\tau\; B(u, Vu, w).$$

Notation 1.5 Finite sets of Delta sentences are denoted Δ (possibly with superscripts) and then

$$\Delta_{\mathtt{r}} :\equiv \{D_{\mathtt{r}} \,|\, D \in \Delta\} \quad \text{and} \quad \widetilde{\Delta} :\equiv \{\widetilde{D} \,|\, D \in \Delta\}.$$

Symbol "D" will stand for some arbitrary-but-fixed (single) Delta sentence and "Δ" for some arbitrary-but-fixed finite set of Delta sentences.

Let $\mathtt{WeZ}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$ and $\mathtt{WeZ}_{\mathtt{m}}^{\exists}$ be the nc-specifying, respectively verifying arithmetics (both based on classical logic) for program extraction by light monotone Dialectica from Chapter 1 of [9]. Both systems are there presented in Natural Deduction (with introduction and elimination rules for conjunction, implication and universal quantifiers,

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and axioms for existential quantifiers) and contain full induction, weakly extensional equality and monotonic elements (inequality and majorizability). The input system $\mathtt{WeZ}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$ also includes the nc universal and existential quantifiers together with the peculiar restriction[3)] concerning the introduction rule for the nc-universal quantifier. The "+" refers to the addition of an nc extension of the usual axiom of quantifier-free choice (see Section 3). The verifying system $\mathtt{WeZ}_{\mathtt{m}}^{\exists}$ contains only the usual, "regular" quantifiers and subsumes full classical logic as a consequence of the inclusion of the full *Comprehension Axiom* scheme

$$\mathtt{CA} :\equiv \exists \Phi^{\tau o}\, \forall x^{\tau}\ [\, \mathtt{at}(\Phi x) \ \leftrightarrow\ B(x)\,]$$

for arbitrary formulas $B(x^{\tau})$ in the language of $\mathtt{WeZ}_{\mathtt{m}}^{\exists}$. This is necessary in the monotone nc context only for the purpose of formally verifying the interpretation of computationally-relevant contractions over formulas with nc quantifiers. In particular, it has no impact on the program-extraction process.

For the simplification of the exposition, in this paper we will always explicitate the addition[4)] of Δ axioms to $\mathtt{WeZ}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$ and, correspondingly, of the $\widetilde{\Delta}$ axioms to $\mathtt{WeZ}_{\mathtt{m}}^{\exists}$. On the contrary, an arbitrary-but-fixed set of axioms $\Pi \equiv \{\forall b B^{\mathtt{nc}}(b)\}$ is still implicitly comprised into $\mathtt{WeZ}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$ (with the r-associated $\widetilde{\Pi} \equiv \{\forall b B(b)\}$ axioms included in $\mathtt{WeZ}_{\mathtt{m}}^{\exists}$ for LMD-verification) but here the "Π axioms" are by default[5)] globally restricted in that all their positive variables have type degree at most 1 (shortly, $\mathrm{dg}^{+}(B) \le 1$).

Notation 1.6 Say *v-proof* short for verifying proof.

The following is a fundamental result of Section 2.3 of [9], obtained as immediate corollary to the so-called "main theorem on bounds extraction by the KNLMD-interpretation", where KNLMD is a short name for the composition of Kuroda negative translation followed by light monotone Dialectica interpretation.

Theorem 1.7 (Uniform bounds by KNLMD-interpretation) *Let* $A_1 \equiv \exists u'\, A^{\mathtt{nc}}(x^{\iota\iota}, k^{\iota}, y^{\delta}, z^{\gamma}, u')$ *be a formula with* x, k, y, z *all its free variables and* $\mathrm{dg}(\gamma) \le 2$. *Let* $s^{(\iota\iota)\iota\delta}$ *be a closed term. There is an* (*light monotone Dialectica*) *algorithm which from a given proof in* $\mathtt{WeZ}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+} + \Delta$ *of*

$$\forall x^{\iota\iota}\, \forall k^{\iota}\ (\forall y \le_{\delta} sxk)\ \exists z^{\gamma}\ A_1(x,k,y,z) \tag{2}$$

yields a closed $\mathtt{t}^{(\iota\iota)\iota\gamma} \in \mathcal{T}_{\mathtt{m}}$ (*a bound for* z, *uniform relative to* y) *and a v-proof in* $\mathtt{WeZ}_{\mathtt{m}}^{\exists} + \widetilde{\Delta}$ *of*

$$\forall x^{\iota\iota}\, \forall k^{\iota}\ (\forall y \le_{\delta} sxk)\ (\exists z \le_{\gamma} \mathtt{t}xk)\ AOneReg(x,k,y,z), \tag{3}$$

where $A_1^{\mathtt{r}}(x,k,y,z) \equiv \exists u'\, A(x,k,y,z,u')$ *is the r-associate of* A_1.

This and the following theorems are substantial variations of the corresponding results established by Kohlenbach (best see [20]), in the sense that, under certain restrictions, majorants and uniform bounds can still be obtained when "quantifier free" is weakened to "all quantifiers are nc". The importance of this extension is that possibly more input specifications can now be treated by the monotone Dialectica, in its light variant.

2 Epsilon-arithmetization with nc quantifiers

The so-called "ε-arithmetization" technique was developed by Kohlenbach initially in [11] (see the second half of page 1243) and later in [15, 16] (see also [20] for a survey). The particle "epsilon" refers to the replacement of, e.g., "$\exists x^{\mathbb{R}} (f(x) =_{\mathbb{R}} 0) \equiv \exists x \forall \varepsilon >_{\mathbb{R}} 0 (|f(x)| <_{\mathbb{R}} \varepsilon)$" by "$\forall \varepsilon >_{\mathbb{R}} 0 \exists x^{\mathbb{R}} (|f(x)| <_{\mathbb{R}} \varepsilon)$" in Analysis, so that key analytical results (such as the intermediate value theorem) become constructively provable (e.g., in the sense of [4]). Kohlenbach uses such logical weakenings of the usual classical analytical theorems in an original way, best see this in [20]. Below we present our immediate nc adaptation of Kohlenbach's definition from [20].

Definition 2.1 With D as in (1) we associate the so-called *ε-weakenings* of $D_{\mathtt{r}}$ and $\widetilde{D}$:

$$D_{\varepsilon} :\equiv \forall w^{\tau} \forall u^{\rho}\ (\exists v \le_{\sigma} ru)\ (\forall \widetilde{w} \le_{\tau} w)\ B(u, v, \widetilde{w}),$$

$$\widetilde{D}_{\varepsilon} :\equiv \forall w^{\tau}\ (\exists V \le_{\rho\sigma} r)\ \forall u^{\rho}\ (\forall \widetilde{w} \le_{\tau} w)\ B(u, Vu, \widetilde{w}).$$

Correspondingly, the ε-weakenings of $\Delta_{\mathtt{r}}$ and $\widetilde{\Delta}$ are $\Delta_{\varepsilon} :\equiv \{D_{\varepsilon} \mid D \in \Delta\}$ and respectively $\widetilde{\Delta}_{\varepsilon} :\equiv \{\widetilde{D}_{\varepsilon} \mid D \in \Delta\}$.

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3) This is originally due to Berger [2] and was explicitated in [9] (see Page 49) as the *variable condition* $\mathtt{VC}_3$. Just for the non-monotone light Dialectica is $\mathtt{VC}_3$ stronger than Berger's restriction. In the monotone setting, $\mathtt{VC}_3$ becomes equivalent to Berger's variable condition.

4) In [9], arbitrary but fixed sets of Delta axioms were implicitly comprised into the system denoted $\mathtt{WeZ}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$.

5) Further local restrictions on the Π axioms will be introduced by need, in Section 4.2.

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Theorem 2.2 (ε-arithmetization of Delta premises) *Let formula A_1 and term s be as in Theorem* 1.7. *There exists an algorithm which from a given proof in* $\mathtt{WeZ}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$ *of*

$$(4) \qquad D \to \forall x^{\iota\iota}\, \forall k^{\iota}\, (\forall y \leq_{\delta} sxk)\, \exists z^{\gamma}\, A_1(x,k,y,z)$$

produces a closed $\mathtt{t}^{(\iota\iota)\iota\gamma} \in \mathcal{T}_{\mathtt{m}}$ *and a v-proof in* $\mathtt{WeZ}_{\mathtt{m}}^{\exists}$ *of*

$$(5) \qquad \widetilde{D}_{\varepsilon} \to \forall x\, \forall k\, (\forall y \leq_{\delta} sxk)\, (\exists z \leq_{\gamma} \mathtt{t}xk)\, A_1^{\mathtt{r}}(x,k,y,z).$$

P r o o f. We reason within $\mathtt{WeZ}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$. Since immediately $\overline{D} \to D$, from (4) we obtain effectively a proof of

$$\overline{D} \to \forall x^{\iota\iota}\, \forall k^{\iota}\, (\forall y \leq_{\delta} sxk)\, \exists z^{\gamma}\, \exists u'\, A^{\mathtt{nc}}(x,k,y,z,u')$$

from which (using just classical logic) we get a proof of

$$\forall x\, \forall k\, (\forall y \leq sxk)\, (\forall V \leq_{\rho\sigma} r)\, (\exists u^{\rho}, w, z, u')\, [B^{\mathtt{nc}}(u, Vu, w) \to A^{\mathtt{nc}}(x,k,y,z,u')]$$

effectively in $\mathtt{WeZ}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$. By a small adaptation of Theorem 1.7 (applied with $\Delta \equiv \emptyset$) we then obtain effectively closed terms $\mathtt{t}, \mathtt{t}' \in \mathcal{T}_{\mathtt{m}}$ and a verifying proof in $\mathtt{WeZ}_{\mathtt{m}}^{\exists}$ of

$$\forall x\, \forall k\, (\forall y \leq sxk)\, (\forall V \leq r)\, \exists u\, \exists u'\, (\exists w \leq_{\tau} \mathtt{t}'xk)\, (\exists z \leq_{\gamma} \mathtt{t}xk)\, [B(u, Vu, w) \to A(x,k,y,z,u')].$$

We now reason within $\mathtt{WeZ}_{\mathtt{m}}^{\exists}$. By intuitionistic logic (although $\mathtt{WeZ}_{\mathtt{m}}^{\exists}$ is fully classical) we further get a proof of

$$\forall x\, \forall k\, [(\exists V \leq r)\, \forall u\, (\forall w \leq \mathtt{t}'xk)\, B(u, Vu, w) \to (\forall y \leq sxk)\, (\exists z \leq \mathtt{t}xk)\, \exists u'\, A(x,k,y,z,u')],$$

which can be intuitionistically weakened to a proof of

$$\forall x \forall k\, [\forall w\, (\exists V \leq r)\, \forall u\, (\forall \widetilde{w} \leq w)\, B(u, Vu, \widetilde{w}) \to (\forall y \leq sxk)\, (\exists z \leq \mathtt{t}xk)\, A_1^{\mathtt{r}}(x,k,y,z)].$$

We have thus obtained effectively a proof in $\mathtt{WeZ}_{\mathtt{m}}^{\exists}$ of (5). □

We now begin to use D_{ε} in the following adaptation of a proposition established by Kohlenbach (e.g., [20]).

Proposition 2.3 $\mathtt{WeZ}_{\mathtt{m}}^{\exists} \vdash \widetilde{D}_{\varepsilon}$ *effectively if and only if* $\mathtt{WeZ}_{\mathtt{m}}^{\exists} \vdash D_{\varepsilon}$.

P r o o f. System $\mathtt{WeZ}_{\mathtt{m}}^{\exists}$ is closed under the full rule of choice:

$$\mathtt{WeZ}_{\mathtt{m}}^{\exists} \vdash \forall x^{\rho} \exists y^{\tau} C(x;y) \Rightarrow \mathtt{WeZ}_{\mathtt{m}}^{\exists} \vdash \exists Y^{\rho\tau} \forall x^{\rho} C(x; Yx).$$

Even without CA, this can be proved using modified realizability with truth like for system $WE\text{–}HA^{\omega}$ in [20]. □

Corollary 2.4 (Elimination of Δ premises by ε-arithmetization) *Let A_1 and s be as in Theorem* 1.7 *and Δ so that* $\mathtt{WeZ}_{\mathtt{m}}^{\exists} \vdash \Delta_{\varepsilon}$. *There is an algorithm which from a proof in* $\mathtt{WeZ}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$ *of*

$$(6) \qquad \Delta \to \forall x^{\iota\iota}\, \forall k^{\iota}\, (\forall y \leq_{\delta} sxk)\, \exists z^{\gamma}\, A_1(x,k,y,z)$$

produces a closed $\mathtt{t}^{(\iota\iota)\iota\gamma} \in \mathcal{T}_{\mathtt{m}}$ *and a v-proof in* $\mathtt{WeZ}_{\mathtt{m}}^{\exists}$ *of*

$$\forall x^{\iota\iota}\, \forall k^{\iota}\, (\forall y \leq_{\delta} sxk)\, (\exists z \leq_{\gamma} \mathtt{t}xk)\, A_1^{\mathtt{r}}(x,k,y,z).$$

P r o o f. In a tedious but straightforward way, the algorithm of Theorem 2.2 can be modified to accept inputs (4) with Δ instead of D. Here Δ is understood as the conjunction of its elements. Correspondingly, the modified algorithm produces outputs (5) with $\widetilde{\Delta}_{\varepsilon}$ instead of $\widetilde{D}_{\varepsilon}$, also understood as the conjunction of its elements. Now Proposition 2.3 applies, combined with the hypothesis that $\mathtt{WeZ}_{\mathtt{m}}^{\exists} \vdash \Delta_{\varepsilon}$. □

As Kohlenbach points out in [16], for many ineffective theorems D of Mathematics (and particularly Numerical Functional Analysis), the corresponding ε-weakenings D_{ε} are (constructively) provable in (subsystems of) $\mathtt{WeZ}_{\mathtt{m}}^{\exists}$.

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Remark 2.5 (Binary König Lemma) Principle $\mathtt{WKL}$, that every infinite binary tree has an infinite branch, can be formalized in more ways as a sentence D for which $\mathtt{WeZ}^{\exists}_{\mathtt{m}} \vdash D_\varepsilon$. See [20] for a largely explained survey. We here only mention Troelstra's classical definition (introduced in [24]) for $\mathtt{WKL}$:

$$\forall f^{\iota\iota}\big(T(f) \wedge \forall x^\iota \exists n^\iota (\mathrm{lth}\, n = x \wedge fn = \mathtt{0}) \to (\exists b \leq_{\iota\iota} \lambda k.\, \mathtt{1})\forall x^\iota (f(\bar{b}x) = \mathtt{0})\big),$$

where $T(f) :\equiv \forall n \forall m (f(n * m) = \mathtt{0} \to fn = \mathtt{0}) \wedge \forall n \forall x (f(n * \langle x \rangle) = \mathtt{0} \to x \leq \mathtt{1})$. Thus, $T(f)$ specifies the fact that f is a $\mathtt{0}, \mathtt{1}$-tree.

Corollary 2.6 (Elimination of $\mathtt{WKL}$ assumptions by ε-arithmetization) *Let the formula A_1 and the term s be as in Theorem* 1.7. *There exists an algorithm which from a given proof*

$$\mathtt{WeZ}^{\exists,\mathtt{nc},\mathtt{c}+}_{\mathtt{m}} \oplus \mathtt{WKL} \ \vdash \ \forall x^{\iota\iota}\, \forall k^\iota\, (\forall y \leq_\delta sxk)\, \exists z^\gamma\, A_1(x,k,y,z)$$

produces a closed $\mathtt{t}^{(\iota\iota)\iota\gamma} \in \mathcal{T}_{\mathtt{m}}$ together with a v-proof

$$\mathtt{WeZ}^{\exists}_{\mathtt{m}} \ \ \vdash \ \ \forall x^{\iota\iota}\, \forall k^\iota\, (\forall y \leq_\delta sxk)\, (\exists z \leq_\gamma \mathtt{t}xk)\, A_1^{\mathtt{r}}(x,k,y,z).$$

Here the addition of $\mathtt{WKL}$ *to* $\mathtt{WeZ}^{\exists,\mathtt{nc},\mathtt{c}+}_{\mathtt{m}}$ *by* $\oplus$ *instead of* $+$ *marks the restriction that* $\mathtt{WKL}$ *shall not be used in any of the proofs of premises of instances of the quantifier-free rule of extensionality/compatibility in the input proof.*

P r o o f. The adjoining of $\mathtt{WKL}$ via $\oplus$ ensures that $\mathtt{WKL}$ can be viewed as an undischarged assumption, rather than an axiom. It can thus be moved as a premise on the right-hand side of $\vdash$. Then Corollary 2.4 applies, modulo Remark 2.5. □

3 nc-Elimination-of-extensionality

We begin this section by mentioning that our $\mathtt{nc}$ weakly-extensional classical arithmetic $\mathtt{WeZ}^{\exists,\mathtt{nc},\mathtt{c}+}_{\mathtt{m}}$ subsumes the following pure-nc *Axiom of Choice* scheme[6]:

$$\mathtt{AC}_{\mathtt{nc}} :\equiv \forall u^\sigma\, \exists v^\tau\, B^{\mathtt{nc}}(u,v) \ \to \ \exists V^{\sigma\tau}\, \forall u\, B^{\mathtt{nc}}(u, V(u)).$$

Definition 3.1 In [20], the logical relation *strong hereditarily extensional equality* between functionals of type ρ, denoted $=^e_\rho$, is defined inductively by $x =^e_\iota y \ :\equiv \ x =_\iota y$ and

$$x =^e_{\sigma\tau} y :\equiv \forall u^\sigma\, \forall v^\sigma [u =^e_\sigma v \to (xu =^e_\tau xv \wedge xu =^e_\tau yv)].$$

Definition 3.2 In [20], the *extensional translation* of formulas $A \longmapsto A_e$ is defined for atomic formulas A by $A_e :\equiv A$ and further by:

$$(A \wedge B)_e :\equiv A_e \wedge B_e, \qquad (A \to B)_e :\equiv A_e \to B_e,$$

$$(\exists x^\rho\, A)_e :\equiv \exists x^\rho\, (x =^e_\rho x \wedge A_e) \quad \text{and}$$

$$(\forall x^\rho\, A)_e :\equiv \forall x^\rho\, (x =^e_\rho x \to A_e).$$

We extend this to $\mathtt{nc}$ *extensional translation*, formulas with $\mathtt{nc}$ quantifiers by (below we could alternatively consider that the - universal - quantifiers involved by $x =^e_\rho x$ are all $\mathtt{nc}$, but this would imply some more technical details in the proofs below)

$$(\exists^{\mathtt{nc}} x^\rho\, A)_e :\equiv \exists^{\mathtt{nc}} x^\rho\, (x =^e_\rho x \wedge A_e) \quad \text{and}$$

$$(\forall^{\mathtt{nc}} x^\rho\, A)_e :\equiv \forall^{\mathtt{nc}} x^\rho\, (x =^e_\rho x \to A_e).$$

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6) This was denoted $\mathtt{AxAC}_{\mathtt{nc}}$ in [9], where also the variant $\mathtt{AxAC}^{\mathtt{cl}}_{\mathtt{nc}} :\equiv \forall u^\sigma\, \exists^{\mathtt{cl}} v^\tau\, B^{\mathtt{nc}}(u,v) \ \to \ \exists^{\mathtt{cl}} V^{\sigma\tau}\, \forall u\, B^{\mathtt{nc}}(u, V(u))$ had been considered, somewhat redundantly, since both $\mathtt{AxAC}_{\mathtt{nc}}$ and $\mathtt{AxAC}^{\mathtt{cl}}_{\mathtt{nc}}$ admit almost the same realizers under $\mathtt{KNLMD}$. Note that their realizers under the (intuitionistic) LMD-translation are more different, though fairly similar as well.

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Let $\mathtt{Z}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$ be the fully extensional variant of $\mathtt{WeZ}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$, obtained by adding to it[7] the full compatibility axiom $x =_\sigma y \to A(x) \to A(y)$ and by restricting $\mathtt{AC}_{\mathtt{nc}}$ to those instances with $\mathrm{dg}(u) + \mathrm{dg}(v) \leq 1$, and moreover such that:

a) If $\mathrm{dg}(u) = 1$, then the pure-nc radical $B^{\mathrm{nc}}(u, v)$ shall actually be an (nc-)quantifier-free formula.

b) If $\mathrm{dg}(u) = 0$, then all the (nc-)quantified variables of B^{nc} shall have type degree at most 1 (hence all variables of $\mathtt{AC}_{\mathtt{nc}}$ shall have type degree at most 1, since in this case also $\mathrm{dg}(V) = 1$ is enforced, regardless of whether $\mathrm{dg}(v) = 0$ or $\mathrm{dg}(v) = 1$; notice that the pure-nc formula B^{nc} appears both positively and negatively inside $\mathtt{AC}_{\mathtt{nc}}$).

Then we can state the following nc adaptation of the simplified variant developed by Kohlenbach in [20] of the so-called elimination-of-extensionality procedure[8]:

Proposition 3.3 (nc-elimination-of-extensionality) *From a proof in* $\mathtt{Z}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$ *of* $A(\underline{a})$ *one can effectively construct a proof in* $\mathtt{WeZ}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$ *of* $\underline{a} =^e \underline{a} \to A_e(\underline{a})$. *Here* $\underline{a}$ *are all the free variables of the conclusion formula* A *and the* Π *axioms of* $\mathtt{WeZ}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$ *are exactly those included in* $\mathtt{Z}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$.

Proof. By induction on the structure of the input proof, following [20]. The $\exists^{\mathrm{nc}}$-axioms and $\forall^{\mathrm{nc}}$-rules can be treated exactly like their isomorphic regular correspondents (i.e., the $\exists$-axioms and $\forall$-rules). No violation of the nc restrictions may appear. The treatment of $\mathtt{AC}_{\mathtt{nc}}$ with $\mathrm{dg}(u) = 1$ is the same as Kohlenbach's, due to the enforced quantifier-free radical. This restriction appears to be necessary because of the argument using a primitive recursive bounded search over such a radical, which therefore must be decidable[9]. For the treatment of the here more general (than those used by Kohlenbach) Π axioms and choice schema $\mathtt{AC}_{\mathtt{nc}}$ with $\mathrm{dg}(u) = 0$ (which can employ an unrestricted pure-nc radical) we need to use the following very important lemma:

Lemma 3.4 (Stability of sentences w.r.t. the e-translation) *Let* $\mathtt{IL}_{\mathtt{m}}^{\exists,\mathtt{nc}}$ *denote the underlying* nc *pure predicate intuitionistic logic of* $\mathtt{Z}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$. *The following hold for arbitrary formulas* A *of the input (specifying) system* $\mathtt{Z}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$:

a) *If* $\mathrm{dg}^-(A) \leq 1$, *then* $\mathtt{IL}_{\mathtt{m}}^{\exists,\mathtt{nc}} \vdash A_e \to A$.

b) *If* $\mathrm{dg}^+(A) \leq 1$, *then* $\mathtt{IL}_{\mathtt{m}}^{\exists,\mathtt{nc}} \vdash A \to A_e$.

Proof. We employ a simultaneous induction on the structure of A to prove that both a) and b) hold at each induction step. The base case, when A has no quantifiers at all, follows immediately by the definition of A_e, which in this case is just identical to A. The induction step for $\wedge$ and $\to$ follows by simple propositional logic, since $(A \wedge B)_e \equiv A_e \wedge B_e$ and also $(A \to B)_e \equiv A_e \to B_e$. For the case of $\to$ we also use that the type-degree restrictions on premise and conclusion are symmetric to each other, with the polarities reversed - hence here we must use the induction hypothesis for both a) and b). For the quantifier steps $A \equiv \forall x A'(x)$, $A \equiv \forall^{\mathrm{nc}} x A'(x)$, $A \equiv \exists x A'(x)$ and $A \equiv \exists^{\mathrm{nc}} x A'(x)$ we use that $\vdash x =^e_\rho x$ for x with $\mathrm{dg}(\rho) \leq 1$, see [20] for this working lemma. What happens is that in $\mathtt{IL}_{\mathtt{m}}^{\exists,\mathtt{nc}}$ one immediately establishes (just by definitions) that $\forall x A'_e(x) \to (\forall x A'(x))_e$, $\forall^{\mathrm{nc}} x A'_e(x) \to (\forall^{\mathrm{nc}} x A'(x))_e$, $(\exists x A'(x))_e \to \exists x A'_e(x)$, $(\exists^{\mathrm{nc}} x A'(x))_e \to \exists^{\mathrm{nc}} x A'_e(x)$. Also the reversed directions hold for $\mathrm{dg}(x) \leq 1$ (using the aforementioned lemma). Thus the more difficult implications needed to prove the quantifier induction step are ensured due to the type-degree restrictions on the actual formula. □

Returning to the proof of Proposition 3.3, our choice axioms restricted for the situation when $\mathrm{dg}(u) = 0$ fall into both[10] categories a) and b) of Lemma 3.4, hence their e-translations are logically equivalent to the original. Since axioms Π fall into category b) of Lemma 3.4, they are at most weakened under the e-translation.

□

[7] Compared to the identically denoted system from [9], here $\mathtt{Z}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$ contains no implicit Δ axioms, but a set of restricted Π axioms, see also the paragraph immediately above Notation 1.6.

[8] This practically useful technique goes back to R. O. Gandy [5, 6], but its first use in connection with the Dialectica interpretation is fully formalized and originally due to Luckhardt [22].

[9] Notice that, unlike the regular system $\mathtt{WeZ}_{\mathtt{m}}^{\exists}$, which subsumes the full comprehension axiom CA and therefore enjoys the decidability property also for formulas with quantifiers, the nc system $\mathtt{WeZ}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$ does not necessarily feature such a property.

[10] Here essential is that $\mathtt{AC}_{\mathtt{nc}}$ belongs to category b), fact which implies its weakening under the e-translation. Nonetheless, due to the symmetry relative to $\to$ of the premise and conclusion from the actual choice axioms, the restrictions necessary for category a) are here equivalent with those necessary for category b).

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From Proposition 3.3, Lemma 3.4 and Corollary 2.4 we get:

Theorem 3.5 (Uniform bounds by $\mathtt{EKNLMD}$-interpretation) *Let* $s^{(\iota\iota)\iota\delta}$ *and* A_1 *be as in Theorem* 1.7 *here moreover such that* $\mathrm{dg}(\delta) \leq 1$ *and* $\mathrm{dg}^-(A^{\mathrm{nc}}) \leq 1$. *Let* Δ *be a set of sentences as defined by* (1) *here further restricted by* $\mathrm{dg}(\sigma) \leq 1$ *and* $\mathrm{dg}^+(B^{\mathrm{nc}}) \leq 1$ *and such that* $\mathtt{WeZ}_{\mathtt{m}}^{\exists} \vdash \Delta_\varepsilon$. *There is an algorithm which from a given proof in* $\mathtt{Z}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+} + \Delta$ *of*

(7) $$\forall x^{\iota\iota}\, \forall k^{\iota}\, (\forall y \leq_\delta sxk)\, \exists z^\gamma\, A_1(x,k,y,z)$$

produces a closed $\mathtt{t}^{(\iota\iota)\iota\gamma} \in \mathcal{T}_{\mathtt{m}}$ *and a v-proof in* $\mathtt{WeZ}_{\mathtt{m}}^{\exists}$ *of*

(8) $$\forall x^{\iota\iota}\, \forall k^{\iota}\, (\forall y \leq_\delta sxk)\, (\exists z \leq_\gamma \mathtt{t}xk)\, A_1^{\mathtt{r}}(x,k,y,z).$$

Thus $\mathtt{t}$ *is a bound for* z, *uniform relative to* y.

P r o o f. Within the fully extensional Natural Deduction system $\mathtt{Z}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$, we can automatically see the axiom $\mathtt{WKL}$ as an open assumption, therefore we can rewrite the proof of (7) as a proof in $\mathtt{Z}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$ of

(9) $$\Delta \to \forall x^{\iota\iota}\, \forall k^{\iota}\, (\forall y \leq_\delta sxk)\, \exists z^\gamma\, A_1(x,k,y,z).$$

Because of the various assumed type-degree restrictions (but not those on γ, τ) the conclusion sentence (9) is placed into the situation a) of Lemma 3.4. By combining this with the outcome of Proposition 3.3 applied to (9), we construct effectively (by e-interpretation) a proof in $\mathtt{WeZ}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+}$ of (9) which, due to the restrictions on $\mathrm{dg}(\gamma)$ and $\mathrm{dg}(\tau)$, fits as an acceptable input (6) to the algorithm of Corollary 2.4. □

Corollary 3.6 (Elimination of $\mathtt{WKL}$ axioms by $\mathtt{EKNLMD}$-interpretation) *Let the term* s *and the formula* A_1 *be as in Theorem* 3.5 *above. There exists an algorithm which from a given proof*

$$\mathtt{Z}_{\mathtt{m}}^{\exists,\mathtt{nc},\mathtt{c}+} + \mathtt{WKL} \vdash \forall x^{\iota\iota}\, \forall k^{\iota}\, (\forall y \leq_\delta sxk)\, \exists z^\gamma\, A_1(x,k,y,z)$$

produces a closed $\mathtt{t}^{(\iota\iota)\iota\gamma} \in \mathcal{T}_{\mathtt{m}}$ *and a verifying proof*

$$\mathtt{WeZ}_{\mathtt{m}}^{\exists} \vdash \forall x^{\iota\iota}\, \forall k^{\iota}\, (\forall y \leq_\delta sxk)\, (\exists z \leq_\gamma \mathtt{t}xk)\, A_1^{\mathtt{r}}(x,k,y,z).$$

P r o o f. In complement to Remark 2.5, we also have that $\mathrm{dg}^+(\mathtt{WKL}) \leq 1$ holds for the usual formulations of $\mathtt{WKL}$. Then Theorem 3.5 applies directly, with $\Delta \equiv \{\mathtt{WKL}\}$. □

4 Model-interpretation with nc-quantifiers

Let $\mathcal{S}^\omega$ denote as usual the full ZFC set-theoretic type structure. Let also $\mathcal{M}^\omega$ denote Bezem's type structure of all strongly majorizable functionals, as usual. Let $\mathtt{BAC}$ denote the following Axiom scheme of Bounded Choice:

(10) $$\forall R^{\rho\sigma}\, [\forall u^\rho\, (\exists v \leq_\sigma Ru)\, C(u,v,R) \to (\exists V \leq_{\rho\sigma} R)\, \forall u\, C(u, Vu, R)],$$

where ρ and σ are arbitrary types and C is an arbitrary formula of $\mathtt{WeZ}_{\mathtt{m}}^{\exists}$ (hence without $\mathtt{nc}$ quantifiers). Since both $\mathcal{S}^\omega$ and $\mathcal{M}^\omega$ are models of the full comprehension axiom $\mathtt{CA}$, it can be easily established (using the relevant literature) that

(11)
a) if $\mathcal{S}^\omega \models \widetilde{\Pi}$, then $\mathcal{S}^\omega \models \mathtt{WeZ}_{\mathtt{m}}^{\exists} + \mathtt{BAC}$,
b) if $\mathcal{M}^\omega \models \widetilde{\Pi}$, then $\mathcal{M}^\omega \models \mathtt{WeZ}_{\mathtt{m}}^{\exists} + \mathtt{BAC}$.

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4.1 Verifications in set-models $\mathcal{M}^\omega$ and $\mathcal{S}^\omega$

Let Δ be a set of sentences as defined by (1). For $R := r$ and $C(u, v, R) :\equiv \forall w B(u, v, w)$ in (10) one obtains

$$\forall u^\rho\ (\exists v \le_\sigma ru)\ \forall w^\tau\ B(u, v, w)\ +\ \mathtt{BAC}\ \vdash\ (\exists V \le_{\rho\sigma} r)\ \forall u^\rho\ \forall w^\tau\ B(u, Vu, w).$$

In consequence, with the Notation 1.5, one has that

$$\Delta_{\mathtt{r}}\ +\ \mathtt{BAC}\ \vdash\ \widetilde{\Delta},$$

hence one can also write

$$\text{(12)}\qquad \mathtt{WeZ}^{\exists}_{\mathtt{m}}\ +\ \Delta_{\mathtt{r}}\ +\ \mathtt{BAC}\ \vdash\ \mathtt{WeZ}^{\exists}_{\mathtt{m}}\ +\ \widetilde{\Delta}.$$

Using (11), we thus conclude that:

a) If $\mathcal{S}^\omega \models \widetilde{\Pi}$ and $\mathcal{S}^\omega \models \Delta_{\mathtt{r}}$, then $\mathcal{S}^\omega \models \mathtt{WeZ}^{\exists}_{\mathtt{m}}(\widetilde{\Pi}) + \widetilde{\Delta}$.

b) If $\mathcal{M}^\omega \models \widetilde{\Pi}$ and $\mathcal{M}^\omega \models \Delta_{\mathtt{r}}$, then $\mathcal{M}^\omega \models \mathtt{WeZ}^{\exists}_{\mathtt{m}}(\widetilde{\Pi}) + \widetilde{\Delta}$.

If instead of a fully syntactic verifying proof, a semantic guarantee that the verification holds in $\mathcal{S}^\omega$ suffices, then relative to Theorem 1.7 we can consider that $\mathtt{WeZ}^{\exists,\mathtt{nc},\mathtt{c}+}_{\mathtt{m}}$ only includes Π axioms for which $\mathcal{S}^\omega \models \widetilde{\Pi}$ and that $\mathcal{S}^\omega \models \Delta_{\mathtt{r}}$. Then the verification goes through in $\mathcal{S}^\omega$. E.g., since WKL is not only a Delta axiom, but is also valid in $\mathcal{S}^\omega$, one thus achieves the admissibility of the full addition of WKL as axiom to the weakly extensional system $\mathtt{WeZ}^{\exists,\mathtt{nc},\mathtt{c}+}_{\mathtt{m}}$ in the (Light) Monotone Dialectica proof mining. Hence one can avoid for WKL the use of ε-arithmetization and elimination-of-extensionality techniques presented in the previous sections (see Corollaries 2.6 and 3.6), if we simply accept a verification in $\mathcal{S}^\omega$. The gain would be that, on one hand we do not need to type-degree restrict $\mathtt{AC}_{\mathtt{nc}}$, yet we can freely add WKL as a full-standing axiom, via a "$+$" and not just a "$\oplus$".

Remark 4.1 If one does not want to avoid the elimination-of-extensionality, then relative to Theorem 3.5 we can consider that $\mathtt{Z}^{\exists,\mathtt{nc},\mathtt{c}+}_{\mathtt{m}}$ only includes Π for which $\mathcal{S}^\omega \models \widetilde{\Pi}$ and that $\mathcal{S}^\omega \models \Delta_{\mathtt{r}}$. Then the verification goes through in $\mathcal{S}^\omega$. One can also replace $\mathcal{S}^\omega$ with $\mathcal{M}^\omega$ as the model of choice.

4.2 Model-interpretation: from $\mathcal{M}^\omega$ to $\mathcal{S}^\omega$

A slightly more complicated situation arises when we are given that $\mathcal{M}^\omega \models \Delta_{\mathtt{r}}$, but we are asked for a verification in $\mathcal{S}^\omega$, instead of $\mathcal{M}^\omega$. Such a demand would be justified in that $\mathcal{S}^\omega$ is a formal model for the usual mathematics, whereas $\mathcal{M}^\omega$ is a non-standard model. On the other hand, as we will exemplify later in the sequel, a number of Delta axioms which play an essential role in the synthesis of polynomial programs out of important proofs in Analysis can be proved valid in $\mathcal{M}^\omega$, but invalid in $\mathcal{S}^\omega$, see [15, 17].

We here assume that $\mathcal{M}^\omega \models \widetilde{\Pi}$ for the set of Π axioms comprised into $\mathtt{WeZ}^{\exists,\mathtt{nc},\mathtt{c}+}_{\mathtt{m}}$. Also let

$$\text{(13)}\qquad \Delta' :\equiv \{D \equiv \forall u' (\exists v' \le r'u') \forall w'\, B^{\mathtt{nc}}(u', v', w')\}$$

be a set of Delta sentences as defined by (1), but further restricted by $\mathrm{dg}^-(D) \le 2$ and $\mathrm{dg}^+(D) \le 1$. Hence in particular $\mathrm{dg}(u'), \mathrm{dg}(w') \le 2$ for the regularly universal quantified variables u' and w'. Also the regularly existential quantified variables v' are restricted by $\mathrm{dg}(v') \le 1$. Moreover, for Δ' we assume that $\mathcal{S}^\omega \models \Delta'_{\mathtt{r}}$.

Lemma 4.2 ([15]) $\mathcal{M}^0 = \mathcal{S}^0$, $\mathcal{M}^1 = \mathcal{S}^1$ *and* $\mathcal{M}^2 \subsetneq \mathcal{S}^2$.

If instead of a fully syntactic verifying proof, a guarantee that the verification holds in the classical set-model $\mathcal{S}^\omega$ suffices, then the following *program* extraction theorem can be established in the spirit of Theorem 4.9 of [15].

Theorem 4.3 *Let* $s^{(\iota\iota)\iota\delta}$ *and* $A_1(\cdot) \equiv \exists u'\, A^{\mathtt{nc}}(\cdot, u')$ *be as in Theorem* 3.5 *here moreover s.t.* $\mathrm{dg}(u') \le 2$. *There is an algorithm which from a proof in* $\mathtt{WeZ}^{\exists,\mathtt{nc},\mathtt{c}+}_{\mathtt{m}} + \Delta + \Delta'$ *of*

$$\text{(14)}\qquad \forall x^{\iota\iota}\, \forall k^\iota\ (\forall y \le_\delta sxk)\ \exists z^\gamma\ A_1(x, k, y, z)$$

produces a closed $\mathtt{t}^{(\iota\iota)\iota\gamma} \in \mathcal{T}_{\mathtt{m}}$ *such that* $\mathcal{S}^\omega$ *models*

$$\text{(15)}\qquad \forall x^{\iota\iota}\, \forall k^\iota\ (\forall y \le_\delta sxk)\ (\exists z \le_\gamma \mathtt{t}xk)\ A_1^{\mathtt{r}}(x, k, y, z).$$

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P r o o f. By Theorem 1.7 one first algorithmically obtains a v-proof of (15) in $\mathtt{WeZ}^{\exists}_{\mathtt{m}} + \widetilde{\Delta} + \widetilde{\Delta'}$. On the other hand, as a particular case of (12) we have that

$$\text{(16)} \qquad \mathtt{WeZ}^{\exists}_{\mathtt{m}} + \Delta_{\mathtt{r}} + \Delta'_{\mathtt{r}} + \mathtt{BAC} \;\vdash\; \mathtt{WeZ}^{\exists}_{\mathtt{m}} + \widetilde{\Delta} + \widetilde{\Delta'}.$$

Because of the restrictions on the type degrees of the variables in $\Delta'_{\mathtt{r}}$, it follows that all these sentences are valid not only in $\mathcal{S}^\omega$ but also in $\mathcal{M}^\omega$ (using Lemma 4.2). Since by assumption $\mathcal{M}^\omega$ is also a model for $\Delta_{\mathtt{r}}$, it follows that $\mathcal{M}^\omega$ is generally a model for $\mathtt{WeZ}^{\exists}_{\mathtt{m}}{+}\Delta_{\mathtt{r}}{+}\Delta'_{\mathtt{r}}{+}\mathtt{BAC}$ - see also [15, 12] for an indication to the easy proof[11)] of $\mathcal{M}^\omega \models \mathtt{WeZ}^{\exists}_{\mathtt{m}}{+}\mathtt{BAC}$. Hence by (16) we can conclude that (15) is valid in $\mathcal{M}^\omega$. Due to the restriction that the type degree of all negative variables of (15) is at most 1 (including $\mathrm{dg}(\delta) \le 1$) and also all positive variables of (15) have type degree at most 2 (including $\mathrm{dg}(\gamma), \mathrm{dg}(u') \le 2$) and using Lemma 4.2, in consequence also $\mathcal{S}^\omega$ is a model of (15). □

Remark 4.4 The theorem above remains valid after the adjoining to $\mathtt{WeZ}^{\exists,\mathtt{nc},\mathtt{c}+}_{\mathtt{m}}$ of a set

$$\text{(17)} \qquad \Pi' :\equiv \{\forall b B^{\mathrm{nc}}(b) \,|\, \mathcal{S}^\omega \models \forall b B(b)\}$$

of axiom sentences to which the same type restrictions as for Δ' apply. We leave this as an easy exercise to the reader, see Section 3.2.2 of [9] for details in the polynomial context.

It is easy to check that the non-standard (i.e., not valid in $\mathcal{S}^\omega$) analytical axiom $\mathtt{F}^-$, defined by[12)]

$$\forall \Phi^{\iota(\iota\iota)\iota} \forall x^{\iota\iota\iota} (\exists y \le_{\iota\iota\iota} x) \forall k^\iota \forall z^{\iota\iota} \forall n^\iota \left[\textstyle\bigwedge_{i <_\iota n} (z i \le_\iota x k i) \to \Phi k (\lambda k^\iota . \mathtt{If}_\iota (k <_\iota n)(z k) \mathtt{0}) \le_\iota \Phi k (y k) \right],$$

can be included into the axiom set Δ, since $\mathtt{F}^-$ has the right Delta shape of (1) and moreover $\mathcal{M}^\omega \models \mathtt{F}^-$ (see Remark 4.17 of [15]). On the other hand, although valid in $\mathcal{M}^\omega$ (see Proposition 4.6 of [15]), the stronger axiom $\mathbf{F}$ defined by

$$\forall \Phi^{\iota(\iota\iota)\iota} \forall x^{\iota\iota\iota} (\exists y \le_{\iota\iota\iota} x) \forall k^\iota (\forall z \le_{\iota\iota} xk) \left[\Phi k z \le_\iota \Phi k (yk) \right]$$

is not directly of Δ shape, because of the type-ι negative universal quantifier expanded from the extensional definition of $z \le_{\iota\iota} yk$. Nevertheless, $\mathbf{F}$ can be made into a Δ axiom by using an nc-universal quantifier instead of the regular quantifier which causes the trouble. Let axiom $\mathtt{F}^{\mathtt{nc}}$ defined as

$$\forall \Phi^{\iota(\iota\iota)\iota} \forall x^{\iota\iota\iota} (\exists y \le_{\iota\iota\iota} x) \forall k^\iota \forall z^{\iota\iota} \left[\forall^{\mathtt{nc}} l^\iota (z l \le_\iota x k l) \to \Phi k z \le_\iota \Phi k (yk) \right]$$

be such an nc-variant of $\mathbf{F}$, which is easily seen to be a Delta sentence in the sense of (1). Since moreover $\mathcal{M}^\omega \models \mathbf{F}$, which is the r-associate of $\mathtt{F}^{\mathtt{nc}}$, i.e., $\mathbf{F} \equiv (\mathtt{F}^{\mathtt{nc}})_{\mathtt{r}}$, it follows that $\mathtt{F}^{\mathtt{nc}}$ is a Δ axiom for Theorem 4.3. Notice that neither $\mathtt{F}^-$, nor $\mathtt{F}^{\mathtt{nc}}$ is a Δ' axiom for Theorem 4.3, because $\mathcal{S}^\omega \not\models \mathtt{F}^-$ and also $\mathcal{S}^\omega \not\models \mathbf{F} \equiv (\mathtt{F}^{\mathtt{nc}})_{\mathtt{r}}$ (see [15] for indications to the proofs of these). One thus obtains, without the elimination-of-extensionality procedure (in comparison, Theorem 4.9 of [15] uses it for admitting $\mathbf{F}$), the following:

Corollary 4.5 (Full admissibility of axioms $\mathtt{F}^-$, $\mathtt{F}^{\mathtt{nc}}$, $\mathtt{WKL}$) *Theorem* 4.3 *on synthesis of* (*only*) *programs by the* KNLMD*-interpretation holds in particular when the proof of* (14) *is given in* $\mathtt{WeZ}^{\exists,\mathtt{nc},\mathtt{c}+}_{\mathtt{m}} + \mathtt{F}^- + \mathtt{F}^{\mathtt{nc}} + \mathtt{WKL}$.

P r o o f. Not only is WKL an $\mathcal{S}^\omega$-valid Delta axiom, but also $\mathrm{dg}^+(\mathtt{WKL}) \le 1$ and $\mathrm{dg}^-(\mathtt{WKL}) \le 2$, see Remark 2.5. □

11) The only novelty here, relative to the corresponding proof in [15], appears to be the inclusion of CA in $\mathtt{WeZ}^{\exists}_{\mathtt{m}}$. But both $\mathcal{S}^\omega$ and $\mathcal{M}^\omega$ are models of CA, which thus poses no problem.

12) Axiom $\mathtt{F}^-$ was invented by Kohlenbach (initially in [14], but see [13] for a more comprehensive exposition) in the context of synthesizing *polynomial* programs out of mathematically strong subsystems of Analysis [15], as a necessary weakening (to finite initial sequences) of the variant $\mathbf{F}$. In contrast to $\mathbf{F}$, the purely syntactical elimination of $\mathtt{F}^-$ goes straightforwardly, with a verification in a strict subsystem of Heyting Arithmetic in all finite types, see [15] for details. Over Peano Arithmetic in all finite types and in the presence of a minimal amount of quantifier-free choice, WKL can be deduced in terms of any of $\mathtt{F}^-$ or $\mathbf{F}$, see Lemmas 4.1 and 4.4 of [14].

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4.3 Comparison with syntactical methods

As this is effectively the case in the actual proof-mining practice (see Kohlenbach's monograph [20]), we here assume to be concerned only with Delta and Π sentences whose r-associates are valid in at least one of $\mathcal{S}^\omega$ or $\mathcal{M}^\omega$.

For the verification in $\mathcal{S}^\omega$, without the addition of Delta axioms whose r-associates are valid in $\mathcal{M}^\omega$ but not in $\mathcal{S}^\omega$, the unique restriction on both Delta and Π axioms is that their r-associates are valid in $\mathcal{S}^\omega$. There are no type-degree restrictions, also not on the conclusion sentence (2). On the contrary, if only-$\mathcal{M}^\omega$-valid Delta axioms (like $\mathtt{F}^-$ or $\mathtt{F}^{\mathtt{nc}}$) need to be added (as in Theorem 4.3), then type-degree restrictions apply both on the r-associates of the Δ' and Π' axioms (defined by (13), respectively (17)) which are valid in $\mathcal{S}^\omega$ (hence also in $\mathcal{M}^\omega$) and on the conclusion sentence (14) (which is the same as (2)). These type-degree restrictions are stronger than the similar ones from Theorem 3.5. Yet in Theorem 4.3 there are no such restrictions on $\mathtt{AC}_{\mathtt{nc}}$, although full extensionality is not allowed. Hence for concrete only-$\mathcal{M}^\omega$-valid Delta axioms which are admissible to both syntactic and model-interpretation techniques (e.g., $\mathtt{F}^-$), the trade-off is between the use of full extensionality/compatibility and the lack of type-degree restrictions on the choice schema.

Otherwise, if the passing through $\mathcal{M}^\omega$ is not needed, hence for the pure-$\mathcal{S}^\omega$ model-interpretation technique, type-degree unrestricted Delta and Π sentences are admissible as axioms and also the conclusion sentence (2) is type-degree unrestricted. Yet, here as well, full extensionality is not allowed in the input proof. Thus, when concrete $\mathcal{S}^\omega$-valid Delta axioms (like WKL) are admissible for both ε-arithmetization + elimination-of-extensionality and model-interpretation, the trade-off appears between the use of full extensionality/compatibility on one hand and on the other hand the lack of type-degree restrictions on the choice and Π axioms and on the conclusion sentence.

Remark 4.6 ($\mathtt{F}^{\mathtt{nc}}$ vs. $\mathbf{F}$) As proved by Kohlenbach in [18], in a fully extensional context, the weaker axiom $\mathtt{F}^-$ actually implies $\mathbf{F}$, hence the purely syntactic elimination of $\mathbf{F}$ is possible via the simulation in terms of $\mathtt{F}^-$, followed by, e.g., Theorem 3.5 (since $\mathtt{F}^-$ is a Delta axiom). On the other hand, the model-interpretation treatment of $\mathbf{F}$ in a weakly extensional setting (via Theorem 4.3) appears technically impossible, due to its unsuitable logical shape. On the contrary, $\mathtt{F}^{\mathtt{nc}}$ is directly admissible to such $\mathcal{M}^\omega$-model-interpretation. We conjecture that $\mathtt{F}^{\mathtt{nc}}$ can be eliminated also via the purely syntactical methods of Sections 2 and 3.

5 Conclusions and future work

Throughout this paper we have seen how the nc quantifiers continue to show useful for program-extraction, at least in theory. They adapt very well to advanced proof-mining techniques like ε-arithmetization, elimination-of-extensionality and model-interpretation. They may be used as an alternative to elimination-of-extensionality for the admissibility of non-standard axioms like $\mathtt{F}^-$, $\mathbf{F}$ and $\mathtt{F}^{\mathtt{nc}}$ in input proofs, with some important gains over the (otherwise drastic) type-degree restrictions which get partly relaxed.

It nevertheless remains essential that one finds a bigger number of practical applications in order to motivate the use of nc quantifiers in actual proof-mining and to justify the extension of their implementation in MinLog.

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