Hybrid functional interpretations

work in progress, jointly with Paulo Oliva

Mircea-Dan Hernest

Comp Logic @ Innsbruck University

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Outline of this Presentation

1. Unifying functional interpretations
2. Multi-modal Linear Logic LL_h^ω
3. A hybrid functional interpretation (Kreisel + Gödel)
4. Simple example applications to Program Extraction
5. Comparison to light Dialectica. The light hybrid interpretation
6. Future work: Extension to other modalities (Howard, Diller-Nahm)
7. Future work: Automated decoration of Modalities (and nc quantifs)

Outline

Recent work on unifying functional interpretations

1. *Unifying functional interpretations*
   Notre Dame Journal of Formal Logic, 47(2):263-290, 2006
2. *Modified realizability interpretation of classical linear logic*
   LICS'2007
3. *Computational interpretations of classical linear logic*
   WoLLIC'07, LNCS 4576:285-296, 2007
4. *An analysis of Gödel's Dialectica interpretation via linear logic*
   Dialectica, 2007
5. *Functional interpretations of linear and intuitionistic logic*
   Submitted for publication, Oct 2007

- *Optimized programs from (non-constructive) proofs by the light (monotone) Dialectica interpretation* - Hernest - PhD thesis

$\Longrightarrow$ *Hybrid Functional Interpretations* - Hernest & Oliva - in preparation

Motivation for the hybrid functional interpretation

Higher-type equality defined as $F \overset{\rho\to\tau}{=} G :\equiv \forall x^\rho (Fx \overset{\tau}{=} Gx)$

Extensionality axiom schema $\mathrm{Ext} :\ x \overset{\rho}{=} y \ \to\ Fx \overset{\tau}{=} Fy$

Problem: no Dialectica witnesses for Ext – universals of $x \overset{\rho}{=} y$

Solution: $!x \overset{\rho}{=} y \ \multimap\ Fx \overset{\tau}{=} Fy$ where $!$ is *Kreisel* modality (written $!_k$)

No realizer for x under MR for $\neg\forall x A_{\mathrm{qf}}(x) \to \exists x \neg A_{\mathrm{qf}}(x)$ Markov Principle
In linear logic this becomes $?\exists x A_{\mathrm{qf}}(x) \multimap \exists x ?A_{\mathrm{qf}}(x)$ which does not have a realizer for the "Kreisel" *whynot* $?_k$ but for the "Gödel" *whynot* $?_g$

- Girard, J.-Y: *Towards a geometry of interaction* – Contemporary Mathematics 92 (1989) – modalities are not cannonical, different modalities can co-exist into a single formal system of linear logic

$\Longrightarrow$ We decorate exponentials ! and ? with *k* or *g* as we find suitable.

But This will not always be possible, due to certain restrictions !!!

Outline

Linear Logic rules for multiplicatives and quantifiers

$$A_{\mathrm{at}}, A_{\mathrm{at}}^{\perp} \quad (\mathrm{id}) \qquad \frac{\Gamma, A \quad \Delta, A^{\perp}}{\Gamma, \Delta}\,(\mathrm{cut}) \qquad \frac{\Gamma}{\pi\{\Gamma\}}\,(\mathrm{per})$$

$$\frac{\Gamma[\gamma_0], A \quad \Gamma[\gamma_1], B}{\Gamma[(z)(\gamma_0, \gamma_1)], A \diamond_z B}\,(\diamond_z) \qquad \frac{\Gamma, A}{\Gamma, A \diamond_{\mathrm{t}} B}\,(\diamond_{\mathrm{t}}) \qquad \frac{\Gamma, B}{\Gamma, A \diamond_{\mathrm{f}} B}\,(\diamond_{\mathrm{f}})$$

$$A \wedge B \;:\equiv\; \forall z^b (A \diamond_z B) \qquad A \vee B \;:\equiv\; \exists z^b (A \diamond_z B)$$

$$A \multimap B \;:\equiv\; A^{\perp} \parr B \qquad (A^{\perp})^{\perp} \;\equiv\; A$$

$$\frac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \otimes B}\,(\otimes) \qquad \frac{\Gamma, A, B}{\Gamma, A \parr B}\,(\parr) \qquad \frac{\Gamma, A}{\Gamma, \forall z A}\,(\forall) \qquad \frac{\Gamma, A[t/z]}{\Gamma, \exists z A}\,(\exists)$$

Co-existent Kreisel and Gödel modalities for LL_h^ω

$$\frac{?_*\Gamma, A}{?_*\Gamma,\ !_*A}\,(!_*) \qquad \frac{\Gamma, A}{\Gamma, ?_*A}\,(?_*) \qquad \frac{\Gamma, ?_*A, ?_*A}{\Gamma, ?_*A}\,(\mathrm{con}_*) \qquad \frac{\Gamma}{\Gamma, ?_*A}\,(\mathrm{wkn}_*)$$

Table: Rules for the exponentials $* \in \{k, g\}$

In mixing both Kreisel's and Gödel's interpretations, we must add also the following restriction on the "Gödel" contraction rule (con_g):

($\star$) if the contraction formula A in (con_g) is refutation relevant, then it must not contain any Kreisel whynot $?_k$ in front of a *computation relevant* subformula, and also no Kreisel bang $!_k$ in front of a *refutation relevant* subformula (terminology will follow)

($\star$) ensures that the interpretation of such formulas A is quantifier-free (hence decidable); ($\star$) is necessary and sufficient for soundness

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Hybrid functional interpretation $\mathrm{LL}^\omega_{\mathrm{h}} \to \mathrm{LL}^\omega$ of formulas

Interpretation of atomic formulas are atomic formulas themselves: $|A_{\mathrm{at}}| :\equiv A_{\mathrm{at}}$ and $|A^\perp_{\mathrm{at}}| :\equiv A^\perp_{\mathrm{at}}$. Assuming we have already defined $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ and $|B|^{\boldsymbol{v}}_{\boldsymbol{w}}$, we define $|A \diamond_z B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} :\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}}$ and

$$
\begin{array}{rclcrcl}
|A \wp B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{v},\boldsymbol{w}} & :\equiv & |A|^{\boldsymbol{fw}}_{\boldsymbol{v}} \wp |B|^{\boldsymbol{gv}}_{\boldsymbol{w}} & \quad & |\exists z A(z)|^{\boldsymbol{x},z}_{\boldsymbol{f}} & :\equiv & |A(z)|^{\boldsymbol{x}}_{\boldsymbol{f}z} \\
|A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{f},\boldsymbol{g}} & :\equiv & |A|^{\boldsymbol{x}}_{\boldsymbol{fv}} \otimes |B|^{\boldsymbol{v}}_{\boldsymbol{gx}} & & |\forall z A(z)|^{\boldsymbol{f}}_{\boldsymbol{y},z} & :\equiv & |A(z)|^{\boldsymbol{f}z}_{\boldsymbol{y}}
\end{array}
$$

We can give different interpretations to the modalities as:

$$
\begin{array}{rclcrcl}
|!_k A|^{\boldsymbol{x}} & :\equiv & !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} & \quad & |!_g A|^{\boldsymbol{x}}_{\boldsymbol{f}} & :\equiv & !|A|^{\boldsymbol{x}}_{\boldsymbol{fx}} \\
|?_k A|_{\boldsymbol{y}} & :\equiv & ?\exists \boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} & & |?_g A|^{\boldsymbol{f}}_{\boldsymbol{y}} & :\equiv & ?|A|^{\boldsymbol{fy}}_{\boldsymbol{y}}
\end{array}
$$

It follows that $|A \multimap B|^{\boldsymbol{f},\boldsymbol{h}}_{\boldsymbol{x},\boldsymbol{w}} \equiv |A|^{\boldsymbol{x}}_{\boldsymbol{fw}} \multimap |B|^{\boldsymbol{hx}}_{\boldsymbol{w}}$ hence

$$
\begin{array}{rclcl}
|A \to_g B|^{\boldsymbol{f},\boldsymbol{h}}_{\boldsymbol{x},\boldsymbol{w}} & \equiv & |!_g A \multimap B|^{\boldsymbol{f},\boldsymbol{h}}_{\boldsymbol{x},\boldsymbol{w}} & \equiv & !_g |A|^{\boldsymbol{x}}_{\boldsymbol{fwx}} \multimap |B|^{\boldsymbol{hx}}_{\boldsymbol{w}} \\
|A \to_k B|^{\boldsymbol{h}}_{\boldsymbol{x},\boldsymbol{w}} & \equiv & |!_k A \multimap B|^{\boldsymbol{h}}_{\boldsymbol{x},\boldsymbol{w}} & \equiv & !_k \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \multimap |B|^{\boldsymbol{hx}}_{\boldsymbol{w}}
\end{array}
$$

Hybrid functional interpretation $\mathrm{LL}_h^\omega \to \mathrm{LL}^\omega$ of proofs

- In $|A|_{\boldsymbol{y}}^{\boldsymbol{x}}$ – *witness variables* are $\boldsymbol{x}$ – *challenge variables* are $\boldsymbol{y}$
- Two-player one-move game – $|A|_{\boldsymbol{y}}^{\boldsymbol{x}}$ is the *adjudication relation*

Theorem (Soundness of Hybrid Interpretation)

If the sequent $A_0(\boldsymbol{z}), \ldots, A_n(\boldsymbol{z})$ *is provable in* LL_h^ω *(*$\boldsymbol{z}$ *are the only free vars) then from its formal proof,* $\mathbf{T}$*-terms* $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n$ *can be extracted such that the translated sequent* $|A_0(\boldsymbol{z})|_{\boldsymbol{x}_0}^{\boldsymbol{a}_0}, \ldots, |A_n(\boldsymbol{z})|_{\boldsymbol{x}_n}^{\boldsymbol{a}_n}$ *is also provable in* LL^ω*, where* $\mathrm{FV}(\boldsymbol{a}_i) \in \{\boldsymbol{z}, \boldsymbol{x}_0, \ldots, \boldsymbol{x}_n\} \backslash \{\boldsymbol{x}_i\}$.

At $\dfrac{\Gamma, ?_g A, ?_g A}{\Gamma, ?_g A}(\mathrm{con}_g)$, due to $(\star)$8 the interpretation of $?_g A$ is decidable

$\boldsymbol{a}$ is witness for $?_g A$ in conclusion where $\boldsymbol{a}(\boldsymbol{y}) := \begin{cases} \boldsymbol{a}_0(\boldsymbol{y}) & \textbf{if } ?_g|A|_{\boldsymbol{y}}^{\boldsymbol{a}_0(\boldsymbol{y})} \\ \boldsymbol{a}_1(\boldsymbol{y}) & \end{cases}$

$\boldsymbol{y}$ fresh challenge, $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$ witnesses for the two $?_g A$ in the premise

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