Hybrid functional interpretations

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Outline of this Presentation

1. Unifying functional interpretations
2. Multi-modal Linear Logic $\mathrm{LL}_{\mathrm{h}}^{\omega}$
3. A hybrid functional interpretation (Kreisel + Gödel)
4. Simple example applications to Program Extraction
5. Comparison to light Dialectica. The light hybrid interpretation
6. Future work: Extension to other modalities (Howard, Diller-Nahm)
7. Future work: Automated decoration of Modalities (and nc quantifs)

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Recent work on unifying functional interpretations

1. *Unifying functional interpretations*
   Notre Dame Journal of Formal Logic, 47(2):263-290, 2006
2. *Modified realizability interpretation of classical linear logic*
   IEEE symposium on Logic in Computer Science (LICS), 2007
3. *Computational interpretations of classical linear logic*
   WoLLIC'07, LNCS 4576:285-296, 2007
4. *An analysis of Gödel's Dialectica interpretation via linear logic*
   To appear in *Dialectica* (journal), hopefully 2008
5. *Functional interpretations of linear and intuitionistic logic*
   Submitted for publication in October 2007

- *Optimized programs from (non-constructive) proofs by the light (monotone) Dialectica interpretation* - Hernest - PhD thesis

$\Longrightarrow$ *Hybrid Functional Interpretations* - Hernest & Oliva - THIS PAPER

Motivation for the hybrid functional interpretation

Higher-type equality defined as $F \overset{\rho\to\tau}{=} G :\equiv \forall x^\rho (Fx \overset{\tau}{=} Gx)$

Extensionality axiom schema $\quad Ext: \; x \overset{\rho}{=} y \;\to\; Fx \overset{\tau}{=} Fy$

Problem: no Dialectica witnesses for Ext, due to universals of $x \overset{\rho}{=} y$

Solution: $!_k (x \overset{\rho}{=} y) \multimap Fx \overset{\tau}{=} Fy$, where $!_k$ is the *"Kreisel modality"*

No realizer for x under MR for *Markov Pr.* $\quad \neg\forall x A_{qf}(x) \to \exists x \neg A_{qf}(x)$
In linear logic this becomes $!?\exists x A_{qf}(x) \multimap \exists x ? A_{qf}(x)$ which has no realizer for the "Kreisel" *whynot* $?_k$ but has for the "Gödel" *whynot* $?_g$

- Girard, J.-Y: *Towards a geometry of interaction* – Contemporary Mathematics 92 (1989) – modalities are not cannonical, different modalities can co-exist into a single formal system of linear logic

$\Longrightarrow$ We decorate exponentials ! and ? with k or g as we find suitable.

BUT: This will not always be possible, due to certain restrictions !!!

Outline

Linear Logic rules for multiplicatives and quantifiers

$$A_{\mathrm{at}}, A_{\mathrm{at}}^{\perp} \quad (\mathrm{id}) \qquad \frac{\Gamma, A \quad \Delta, A^{\perp}}{\Gamma, \Delta}\,(\mathrm{cut}) \qquad \frac{\Gamma}{\pi\{\Gamma\}}\,(\mathrm{per})$$

$$\frac{\Gamma[\gamma_0], A \quad \Gamma[\gamma_1], B}{\Gamma[(z)(\gamma_0, \gamma_1)], A \diamond_z B}\,(\diamond_z) \qquad \frac{\Gamma, A}{\Gamma, A \diamond_{\mathrm{t}} B}\,(\diamond_{\mathrm{t}}) \qquad \frac{\Gamma, B}{\Gamma, A \diamond_{\mathrm{f}} B}\,(\diamond_{\mathrm{f}})$$

$$A \wedge B \;:\equiv\; \forall z^b (A \diamond_z B) \qquad A \vee B \;:\equiv\; \exists z^b (A \diamond_z B)$$

$$A \multimap B \;:\equiv\; A^{\perp} \parr B \qquad (A^{\perp})^{\perp} \;\equiv\; A$$

$$\frac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \otimes B}\,(\otimes) \qquad \frac{\Gamma, A, B}{\Gamma, A \parr B}\,(\parr) \qquad \frac{\Gamma, A}{\Gamma, \forall z A}\,(\forall) \qquad \frac{\Gamma, A[t/z]}{\Gamma, \exists z A}\,(\exists)$$

Coexistent Kreisel and Gödel modalities for LL_h^ω

$$\frac{?_*\Gamma, A}{?_*\Gamma,\ !_*A}\,(!_*) \qquad \frac{\Gamma, A}{\Gamma, ?_*A}\,(?_*) \qquad \frac{\Gamma, ?_*A, ?_*A}{\Gamma, ?_*A}\,(\mathrm{con}_*) \qquad \frac{\Gamma}{\Gamma, ?_*A}\,(\mathrm{wkn}_*)$$

Table: Rules for the exponentials $* \in \{k, g\}$

In mixing both Kreisel's and Gödel's interpretations, we must add also the following restriction on the "Gödel" contraction rule (con_g):

- $(\star)$ if the contraction formula A in (con_g) is *computationally relevant*, then it must not contain any Kreisel whynot $?_k$ in front of a computationally relevant subformula, and also no Kreisel bang $!_k$ in front of a *refutation relevant* subformula (terminology will follow)

$(\star)$ ensures that the interpretation of such formulas A is quantifier-free (hence decidable); $(\star)$ is necessary and sufficient for soundness of program extraction via Kreisel+Gödel hybrid functional interpretation

Outline

Hybrid functional interpretation $\mathrm{LL}^\omega_{\mathrm{h}} \mapsto \mathrm{LL}^\omega$ of *formulas*

Interpretation of atomic formulas are atomic formulas themselves: $|A_{at}| :\equiv A_{at}$ and $|A_{at}^\perp| :\equiv A_{at}^\perp$. Assuming we have already defined $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ and $|B|^{\boldsymbol{v}}_{\boldsymbol{w}}$, we define $|A \diamond_z B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} :\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}}$ and

$$|A \mathbin{⅋} B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{y},\boldsymbol{w}} :\equiv |A|^{\boldsymbol{f}\boldsymbol{w}}_{\boldsymbol{y}} \mathbin{⅋} |B|^{\boldsymbol{g}\boldsymbol{y}}_{\boldsymbol{w}} \qquad |\exists z\, A(z)|^{\boldsymbol{x},z}_{\boldsymbol{f}} :\equiv |A(z)|^{\boldsymbol{x}}_{\boldsymbol{f}z}$$

$$|A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{f},\boldsymbol{g}} :\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{v}} \otimes |B|^{\boldsymbol{v}}_{\boldsymbol{g}\boldsymbol{x}} \qquad |\forall z\, A(z)|^{\boldsymbol{f}}_{\boldsymbol{y},z} :\equiv |A(z)|^{\boldsymbol{f}z}_{\boldsymbol{y}}$$

“Kreisel” $\neq$ “Gödel” interpretations to co-modalities $?/!$ are given:

$$|!_k A|^{\boldsymbol{x}} :\equiv\ !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \qquad |!_g A|^{\boldsymbol{x}}_{\boldsymbol{f}} :\equiv\ !|A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}}$$

$$|?_k A|_{\boldsymbol{y}} :\equiv\ ?\exists \boldsymbol{x} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \qquad |?_g A|^{\boldsymbol{f}}_{\boldsymbol{y}} :\equiv\ ?|A|^{\boldsymbol{f}\boldsymbol{y}}_{\boldsymbol{y}}$$

It follows that $|A \multimap B|^{\boldsymbol{f},\boldsymbol{h}}_{\boldsymbol{x},\boldsymbol{w}} \equiv |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{w}} \multimap |B|^{\boldsymbol{h}\boldsymbol{x}}_{\boldsymbol{w}}$ hence

$$|A \rightarrow_g B|^{\boldsymbol{f},\boldsymbol{h}}_{\boldsymbol{x},\boldsymbol{w}} \equiv |!_g A \multimap B|^{\boldsymbol{f},\boldsymbol{h}}_{\boldsymbol{x},\boldsymbol{w}} \equiv\ !|A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{w}\boldsymbol{x}} \multimap |B|^{\boldsymbol{h}\boldsymbol{x}}_{\boldsymbol{w}}$$

$$|A \rightarrow_k B|^{\boldsymbol{h}}_{\boldsymbol{x},\boldsymbol{w}} \equiv |!_k A \multimap B|^{\boldsymbol{h}}_{\boldsymbol{x},\boldsymbol{w}} \equiv\ !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \multimap |B|^{\boldsymbol{h}\boldsymbol{x}}_{\boldsymbol{w}}$$

Hybrid functional interpretation $\mathsf{LL}^\omega_{\mathsf{h}} \mapsto \mathsf{LL}^\omega$ of *proofs*

- In $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ – *witness variables* are $\boldsymbol{x}$ – *challenge variables* are $\boldsymbol{y}$
- Two-player one-move game – $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ is the *adjudication relation*

Theorem (Soundness of Hybrid Interpretation)

If the sequent $A_n(\boldsymbol{z}), \ldots, A_0(\boldsymbol{z})$ is provable in $\mathsf{LL}^\omega_{\mathsf{h}}$ [$\boldsymbol{z}$ are all the free variables] then from its formal proof, terms $\boldsymbol{a}_n, \ldots, \boldsymbol{a}_0$ in Gödel's T can be extracted such that the translated sequent $|A_n(\boldsymbol{z})|^{\boldsymbol{a}_n}_{\boldsymbol{y}_n}, \ldots, |A_0(\boldsymbol{z})|^{\boldsymbol{a}_0}_{\boldsymbol{y}_0}$ is provable in LL^ω, where $\mathsf{FV}(\boldsymbol{a}_i) \in \{\boldsymbol{z}, \boldsymbol{y}_n, \ldots, \boldsymbol{y}_0\} \backslash \{\boldsymbol{y}_i\}$.

Treatment of (con_g): $\dfrac{\Gamma, ?_g A, ?_g A}{\Gamma, ?_g A} \mapsto \dfrac{|\Gamma|^{\boldsymbol{a}'}_{\boldsymbol{y}'}, ?_g|A|^{\boldsymbol{a}_1(\boldsymbol{y}_1)}_{\boldsymbol{y}_1}, ?_g|A|^{\boldsymbol{a}_0(\boldsymbol{y}_0)}_{\boldsymbol{y}_0}}{|\Gamma|^{\boldsymbol{a}'}_{\boldsymbol{y}'}, ?_g|A|^{\boldsymbol{a}(\boldsymbol{y})}_{\boldsymbol{y}}}$**, where**

- $\boldsymbol{y}$ are fresh challenge variables which replace both $\boldsymbol{y}_1$ and $\boldsymbol{y}_0$
- $\boldsymbol{a}$ equalizes $\boldsymbol{a}_1$ and $\boldsymbol{a}_0$, i.e., $\boldsymbol{a}(\boldsymbol{y}) := \begin{cases} \boldsymbol{a}_0(\boldsymbol{y}) & \textbf{if } ?_g|A|^{\boldsymbol{a}_0(\boldsymbol{y})}_{\boldsymbol{y}} \\ \boldsymbol{a}_1(\boldsymbol{y}) & \textbf{otherwise} \end{cases}$

Due to $(\star)$, formula $?_g|A|^{\boldsymbol{a}_0(\boldsymbol{y})}_{\boldsymbol{y}}$ is quantifier-free, hence decidable!

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