From formal languages to dynamical systems, C^*-algebras and K-theory

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March 2009

Formal languages

Let $\mathbb{A}$ be a finite set - an alphabet.
A finite string

$$a_1 a_2 \ldots a_n$$

of elements from $\mathbb{A}$ is a *word*.
A collection $\mathbb{W}(S)$ of words is a *formal language*.

In this talk we consider only formal languages which are

- infinite, i.e. $\mathbb{W}(S)$ contains infinitely many words,
- prolongable, i.e. $\forall u \in \mathbb{W}(S)\ \exists a, b \in \mathbb{A} : \ aub \in \mathbb{W}(S)$, and
- factorial, i.e. $u \subseteq v \in \mathbb{W}(S) \Rightarrow u \in \mathbb{W}(S)$.

From formal languages to subshifts

Given a formal language $\mathbb{W}(S)$, set

$$S = \left\{ (a_i)_{i \in \mathbb{Z}} \in \mathbb{A}^{\mathbb{Z}} : \ a_{j+1}a_{j+2}a_{j+3}\dots a_{j+k} \in \mathbb{W}(S) \ \forall j \in \mathbb{Z} \ \forall k \in \mathbb{N} \right\}.$$

$\mathbb{A}^{\mathbb{Z}}$ is a compact Hausdorff space in the product topology;

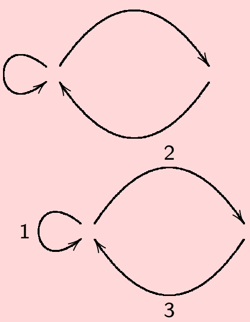
$$d(a,b) = \sum_{i \in \mathbb{Z}} 2^{-|i|} \begin{cases} 0, & \text{when } a_i = b_i \\ 1, & \text{otherwise} \end{cases}$$

S is a *subshift*, i.e. S is closed and $\sigma(S) = S$ where σ is the shift: $\sigma(x)_i = x_{i+1}$

This above correspondence is a bijection between subshifts of $\mathbb{A}^{\mathbb{Z}}$ and formal languages in the alphabet $\mathbb{A}$.

D. Lind, B. Marcus, 'Symbolic Dynamics and Coding', Cambridge University Press, 1995.

Graphs and subshifts of finite type



$\ldots 31112323111112323231231111123 \ldots$

All bi-infinite sequences of 1,2,3 where 13,22,21,33 do not occur; a shift *of finite type* (This particular one is called *the golden mean shift*.)

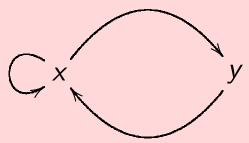
Conjugacy of subshifts

Two subshifts S_1 and S_2 are *conjugate* when there is a homeomorphism

$$\varphi : S_1 \to S_2$$

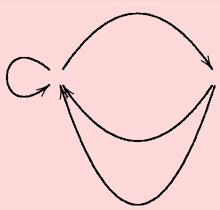
such that $\sigma \circ \varphi = \varphi \circ \sigma$.

The sequences in x and y with the property that yy does not occur constitute a subshift conjugate to the subshift from the previous screen:



Conjugacy means that the subshifts are the same, even if they appear to be different.

The adjacency matrix and conjugacy of SFTs



Adjacency matrix $= \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$

Strong shift equivalence:

$$A \simeq B : \quad A = R_0S_0,\ S_0R_0 = R_1S_1,\ S_1R_1 = R_2S_2 = \dots$$
$$\dots S_{n-1}R_{n-1} = R_nS_n,\ S_nR_n = B$$

Shift equivalence:

$$A \sim B : \quad AR = RB,\ SA = BS,\ A^n = RS,\ B^n = SR.$$

$$A \simeq B \ \Rightarrow\ A \sim B$$

Flow equivalence

Suspension: $\Sigma(S) = S \times \mathbb{R} / (x, t+1) \sim (\sigma(x), t)$
The suspension flow:

$$\sigma_s[x,t] = [x, t+s], \quad s \in \mathbb{R}.$$

Two subshifts S_1 and S_2 are *flow equivalent* when there is a homeomorphism $\varphi : \Sigma(S_1) \to \Sigma(S_2)$ such that

$$\sigma_s \circ \varphi = \varphi \circ \sigma_s$$

for all s.
Two shifts of finite type, given by matrices A and B, are flow equivalent if and only if only

$$\mathbb{Z}^n/(1-A^t)(\mathbb{Z}^n) \simeq \mathbb{Z}^m/(1-B^t)(\mathbb{Z}^m)$$

and $\mathrm{Det}(1-A^t)$, $\mathrm{Det}(1-B^t)$ have the same sign.

D. Lind, B. Marcus, 'Symbolic Dynamics and Coding', Cambridge University Press, 1995.

Flow equivalence from C^*-algebras

Cuntz and Krieger : $A \longrightarrow \mathcal{O}_A$

A and B flow equivalent $\Rightarrow \mathcal{O}_A \simeq \mathcal{O}_B$.

K-homology : $K^0(\mathcal{O}_A) \simeq \mathbb{Z}^n/(1 - A^t)(\mathbb{Z}^n)$

(K. Matsumoto) : General subshift $S \longrightarrow$ a C^*-algebra $\mathcal{O}_S$

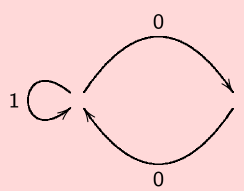
S_1 flow equivalent to $S_2 \Rightarrow \mathcal{O}_{S_1} \simeq \mathcal{O}_{S_2}$.

All isomorphism invariants of $\mathcal{O}_S$ can be used to separate flow equivalence classes of subshift.

J. Cuntz and W. Krieger, 'A class of C^*-algebras and topological Markov chains', Inv. Math. 56(1980), 251-268.
K. Matsumoto, 'On C^*-algebras associated with subshifts', Internat. J. Math. 8 (1997) + a series of more recent papers.

See also the papers by Toke M. Carlsen.

Sofic shifts and regular languages



The language in the alphabet $\{0, 1\}$ which disallows the words $10000\ldots0001$, when the number of 0's is odd.

The even shift - not of finite type.

Using Matsumoto's C^*-algebras it can be shown that the even shift is not flow equivalent to a subshift of finite type.

From equivalence relations to algebras

F - a finite set, $\sim$ an equivalence relation on F.

$$R = \{(x,y) \in F \times F : \ x \sim y\}$$

the graph of $\sim$.

$f, g \in C(R)$. Define $f \star g \in C(R)$ such that

$$f \star g(x,y) = \sum_z f(x,z)g(z,y)$$

$$C(R) \simeq M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_N}$$

N is the number of equivlance classes, n_i is the number of elements in the i'th equivalence class.

From subshifts to algebras

Let $x = (x_i)_{i \in \mathbb{Z}}, y = (y_i)_{i \in \mathbb{Z}} \in S$. Set $x \sim y$ when $\exists N \in \mathbb{N} : \ x_i = y_i, \ i \geq N$.

Give $R = \{(x,y) \in S \times S : \ x \sim y\}$ a locally compact Hausdorff topology such that

$$\{y \in S : \ (x,y) \in C\} \ \text{is finite}$$

for all $x \in S$ and all compact subsets $C \subseteq R$.
The relative topology inherited from $S \times S$ does not work!!

$f, g \in C_c(R)$: $f \star g(x,y) = \sum_z f(x,z) g(z,y)$.

K. Thomsen, 'Semi etale groupoids and applications', arXives, 15/1 '09

Orbit equivalence for minimal Cantor systems

Orbit equivalence: $(X, \varphi) \sim (Y, \psi)$ when there is a homeoprhism $\pi : X \to Y$ such that

$$\pi\left(\left\{\varphi^k(x) :\ k \in \mathbb{Z}\right\}\right) = \left\{\psi^k(\pi(x)) :\ k \in \mathbb{Z}\right\}$$

for all $x \in X$

$$(X, \varphi) \ \rightarrow \ \text{a } C^*\text{-algebra} \ \rightarrow \ \text{its K-theory}$$

is a complete invariant for orbit equivalence minimal homeomorphisms of the Cantor sets,
In particular, for minimal subshifts = bi-infinite sequences with the property that all occuring words occur infinitely often with 'bounded gaps'..

All papers by Giordano, Putnam and Skau, starting with T. Giordano, I. Putnam, C. Skau, 'Topological orbit equivalence and C^*-crossed products', J. Reine u. Ang. Math. 469 (1995),51-111.