Solutions of Generalized Recursive Metric-Space Equations

(Extended Abstract)*

Lars Birkedal, Kristian Støvring, and Jacob Thamsborg

IT University of Copenhagen†

Abstract

It is well known that one can use an adaptation of the inverse-limit construction to solve recursive equations in the category of complete ultrametric spaces. We show that this construction generalizes to a large class of categories with metric-space structure on each set of morphisms: the exact nature of the objects is less important. In particular, the construction immediately applies to categories where the objects are ultrametric spaces with 'extra structure', and where the morphisms preserve this extra structure. The generalization is inspired by classical domain-theoretic work by Smyth and Plotkin. Our primary motivation for solving generalized recursive metric-space equations comes from recent and ongoing work on Kripke-style models in which the sets of worlds must be recursively defined.

For many of the categories we consider, there is a natural subcategory in which each set of morphisms is required to be a compact metric space. Our setting allows for a proof that such a subcategory always inherits solutions of recursive equations from the full category.

As another application, we present a construction that relates solutions of generalized domain equations in the sense of Smyth and Plotkin to solutions of equations in our class of categories.

1 Introduction

Smyth and Plotkin [17] showed that in the classical inverse-limit construction of solutions to recursive domain equations, what matters is not that the *objects* of the category under consideration are domains, but that the sets of *morphisms* between objects are domains. In this work we show that, in the case of ultrametric spaces, the standard construction of solutions to recursive metric-space equations [5, 10] can be similarly generalized to a large class of categories with metric-space structure on each set of morphisms.

The generalization in particular allows one to solve recursive equations in categories where the objects are ultrametric spaces with some form of additional structure, and where the morphisms preserve this additional structure. Our main motivation for solving equations in such categories comes from recent and ongoing work in denotational semantics by the authors and others [7, 15]. There, solutions to such equations are used in order to construct Kripke models over recursively defined worlds: a novel approach that allows one to give semantic models of predicates and relations over languages with dynamically allocated, higher-order store. See Birkedal et al. [8] for examples of such applications.

For many of the categories we consider, there is a natural variant, indeed a subcategory, in which each set of morphisms is required to be a compact metric space [2, 9]. Our setting allows for a general proof that such a subcategory inherits solutions of recursive equations from the full category. Otherwise put, the problem of solving recursive equations in such a 'locally compact' subcategory is, in a certain sense, reduced to the similar problem for the full category. The fact that one can solve recursive equations in a category of compact ultrametric spaces [9] arises as a particular instance. (For various

*See the full article for proofs and further details [8].

†Rued Langgaards Vej 7, 2300 Copenhagen S, Denmark.

applications of compact metric spaces in semantics, see the references in the introduction to van Breugel and Warmerdam [9].)

As another application, we present a construction that relates solutions of generalized domain equations in the sense of Smyth and Plotkin to solutions of equations in our class of categories. This construction generalizes and improves an earlier one due to Baier and Majster-Cederbaum [6].

The key to achieving the right level of generality in the results lies in inspiration from enriched category theory. We shall not refer to general enriched category theory below, but rather present the necessary definitions in terms of metric spaces. The basic idea is, however, that given a cartesian category V (or more generally, a monoidal category), one considers so-called V-categories, in which the 'hom-sets' are in fact objects of V instead of sets, and where the 'composition functions' are morphisms in V.

Other related work. The idea of considering categories with metric spaces as hom-sets has been used in earlier work [9, 14]. Rutten and Turi [14] show existence and uniqueness of fixed points in a particular category of (not necessarily ultrametric) metric spaces, but with a proof where parts are more general. In other work, van Breugel and Warmerdam [9] show uniqueness for a more general notion of categories than ours, again not requiring ultrametricity. Neither of these articles contain a theorem about existence of fixed points for a general class of 'metric-enriched' categories (as in our Theorem 3.1), nor a general theorem about fixed points in locally compact subcategories (Theorem 4.1.)

Alessi et al. [3] consider solutions to *non-functorial* recursive equations in certain categories of metric spaces, i.e., recursive equations whose solutions cannot necessarily be described as fixed-points of functors. In contrast, we only consider *functorial* recursive equations in this work.

Wagner [18] gives a comprehensive account of a generalized inverse limit construction that in particular works for categories of metric spaces and categories of domains. Another such construction has recently been given by Kostanek and Waszkiewicz [11]. Our generalization is in a different direction, namely to categories where the hom-sets are metric spaces. We do not know whether there is a common generalization of our work and Wagner's work; in this work we do not aim for maximal generality, but rather for a level of generality that seems right for our applications [8].

2 Ultrametric spaces

We first recall some basic definitions and properties about metric spaces [13, 16]. A metric space (X,d) is 1-*bounded* if $d(x,y) \leq 1$ for all x and y in X. We shall only work with 1-bounded metric spaces. One advantage of doing so is that one can define coproducts and general products of such spaces; alternatively, one could have allowed infinite distances.

An *ultrametric space* is a metric space (X,d) that satisfies the 'ultrametric inequality' $d(x,z) \leq \max(d(x,y),d(y,z))$ and not just the weaker triangle inequality (where one has $+$ instead of max on the right-hand side). It might be helpful to think of the function d of an ultrametric space (X,d) not as a measure of (euclidean) distance between elements, but rather as a measure of the degree of similarity between elements.

Let CBUlt be the category with complete, 1-bounded ultrametric spaces as objects and non-expansive (i.e., non-distance-increasing) functions as morphisms [5]. This category is cartesian closed [16]; here one needs the ultrametric inequality. The terminal object is the one-point space. Binary products are defined in the natural way: the distance between two pairs of elements is the maximum of the two pointwise distances. The exponential $A \to B$, sometimes written B^A, has the set of non-expansive functions from A to B as the underlying set, and the 'sup'-metric $d_{A\to B}$ as distance function: $d_{A\to B}(f,g) = \sup\{d_B(f(x),g(x)) \mid x \in A\}$. For both products and exponentials, limits are pointwise. It follows from the cartesian closed structure that the function $C^B \times B^A \to C^A$ given by composition is non-expansive; this fact is needed in several places below.

2.1 M-categories

The basic idea of this work is to generalize a theorem about a particular category of metric spaces to a theorem about more general categories where each hom-set is an ultrametric space. In analogy with the O-categories of Smyth and Plotkin (O for 'order' or 'ordered') we call such categories M-categories.

Definition 2.1. An *M-category* is a category $\mathscr{C}$ where each hom-set $\mathscr{C}(A,B)$ is equipped with a distance function turning it into a non-empty, complete, 1-bounded ultrametric space, and where each composition function $\circ : \mathscr{C}(B,C) \times \mathscr{C}(A,B) \to \mathscr{C}(A,C)$ is non-expansive with respect to these metrics. (Here the domain of such a composition function is given the product metric.)

Notice that the hom-sets of an M-category are required to be *non-empty* metric spaces. This restriction allows us to avoid tedious special cases in the results below since the proofs depend on Banach's fixed-point theorem.

The simplest example of an M-category is the category $\mathsf{CBUlt}_{\mathrm{ne}}$ of non-empty, 1-bounded, complete ultrametric spaces and non-expansive maps. Here the distance function on each hom-set $\mathsf{CBUlt}_{\mathrm{ne}}(A,B)$ is given by $d(f,g) = \sup\{d_B(f(x),g(x)) \mid x \in A\}$. The category $\mathsf{CBUlt}_{\mathrm{ne}}$ is cartesian closed since CBUlt is: it suffices to verify that CBUlt-products of non-empty metric spaces are non-empty, and similarly for exponentials.

Let $\mathscr{C}$ be an M-category. A functor $F : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ is *locally contractive* if there exists some $c < 1$ such that $d(F(f,g),F(f',g')) \le c \cdot \max(d(f,f'),d(g,g'))$ for all f, f', g, and g'. Notice that the same c must work for all hom-sets of $\mathscr{C}$.

3 Solving recursive equations

Let $\mathscr{C}$ be an M-category. We consider mixed-variance functors $F : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ on $\mathscr{C}$ and recursive equations of the form $X \cong F(X,X)$. In other words, given such an F we seek a fixed point of F up to isomorphism.

Covariant endofunctors on $\mathscr{C}$ are a special case of mixed-variance functors. It would in some sense suffice to study covariant functors: if $\mathscr{C}$ is an M-category, then so are $\mathscr{C}^{\mathrm{op}}$ (with the same metric on each hom-set as in $\mathscr{C}$) and $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ (with the product metric on each hom-set), and it is well-known how to construct a 'symmetric' endofunctor on $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$ from a functor such as F above. We explicitly study mixed-variance functors since the proof of the existence theorem below would in any case involve an M-category of the form $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$. As a benefit we directly obtain theorems of the form useful in applications. For example, for the existence theorem we are interested in completeness conditions on $\mathscr{C}$, not on $\mathscr{C}^{\mathrm{op}} \times \mathscr{C}$.

3.1 Uniqueness of solutions

Our results below depend on the assumption that the given functor F on $\mathscr{C}$ is locally contractive. One easy consequence of this assumption is that, unlike in the domain-theoretic setting [17], there is at most one fixed point of F up to isomorphism.

Theorem 3.1. *Let $F : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ be a locally contractive functor on an M-category $\mathscr{C}$, and assume that $i : F(A,A) \to A$ is an isomorphism. Then the pair (i, i^{-1}) is a* bifree algebra *for F in the following* *sense: for all objects B of $\mathscr{C}$ and all morphisms $f : F(B,B) \to B$ and $g : B \to F(B,B)$, there exists a unique pair of morphisms $(k : B \to A, h : A \to B)$ such that $h \circ i = f \circ F(k,h)$ and $i^{-1} \circ k = F(h,k) \circ g$:*

$$\begin{array}{ccc} F(A,A) & \underset{F(h,k)}{\overset{F(k,h)}{\rightleftarrows}} & F(B,B) \\ i^{-1}\uparrow\downarrow i & & g\uparrow\downarrow f \\ A & \underset{k}{\overset{h}{\rightleftarrows}} & B \end{array}$$

In particular, A is the unique fixed point of F up to isomorphism.

3.2 Existence of solutions

In the existence theorem for fixed points of contractive functors, the M-category $\mathscr{C}$ will be assumed to satisfy a certain completeness condition involving limits of ω^{op}-chains. Since there are different M-categories satisfying more or less general variants of this condition, it is convenient to present the existence theorem in a form that lists a number of successively weaker conditions.

An *increasing Cauchy tower* is a diagram

$$A_0 \underset{g_0}{\overset{f_0}{\rightleftarrows}} A_1 \underset{g_1}{\overset{f_1}{\rightleftarrows}} \cdots \underset{g_{n-1}}{\overset{f_{n-1}}{\rightleftarrows}} A_n \underset{g_n}{\overset{f_n}{\rightleftarrows}} \cdots$$

where $g_n \circ f_n = id_{A_n}$ for all n, and where $\lim_{n\to\infty} d(f_n \circ g_n, id_{A_{n+1}}) = 0$. Notice that this definition only makes sense for M-categories. The M-category $\mathscr{C}$ has *inverse limits of increasing Cauchy towers* if for every such diagram, the sub-diagram containing only the arrows g_n has a limit. (This subdiagram is, incidentally, an ω^{op}-chain of morphisms that are split epi, i.e., have a left inverse.)

Theorem 3.2. *Assume that the M-category $\mathscr{C}$ satisfies any of the following (successively weaker) conditions:*

1. *$\mathscr{C}$ is complete.*
2. *$\mathscr{C}$ has a terminal object and limits of ω^{op}-chains.*
3. *$\mathscr{C}$ has a terminal object and limits of ω^{op}-chains of split epis.*
4. *$\mathscr{C}$ has a terminal object and inverse limits of increasing Cauchy towers.*

Then every locally contractive functor $F : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ on $\mathscr{C}$ has a unique fixed point up to isomorphism.

4 Locally compact subcategories of M-categories

The condition in Theorem 3.2 that involves Cauchy towers is included in order to accommodate categories where the hom-sets are compact ultrametric spaces [2, 9]: one example is the full subcategory $\mathsf{KBUlt}_{\mathsf{ne}}$ of *compact*, non-empty, 1-bounded ultrametric spaces. This subcategory is merely the simplest example of a full, 'locally compact' subcategory of an M-category. Such a subcategory always inherits fixed points of functors from the full category:

Theorem 4.1. *Assume that $\mathscr{C}$ is an M-category with a terminal object and limits of ω^{op}-chains of split epis. Let I be an arbitrary object of $\mathscr{C}$, and let $\mathscr{D}$ be the full subcategory of $\mathscr{C}$ consisting of those objects A such that the metric space $\mathscr{C}(I,A)$ is compact. $\mathscr{D}$ is an M-category with limits of increasing Cauchy towers, and hence every locally contractive functor $F : \mathscr{D}^{\mathrm{op}} \times \mathscr{D} \to \mathscr{D}$ has a unique fixed point up to isomorphism.*

For a monoidal closed $\mathscr{C}$, the tensor unit is an appropriate choice of I. In particular, taking $\mathscr{C}$ to be $\mathsf{CBUlt}_{\mathrm{ne}}$ and I to be one-point metric space, one obtains:

Corollary 4.2 ([9]). *Every locally contractive functor $F : {\mathsf{KBUlt}_{\mathrm{ne}}}^{\mathrm{op}} \times \mathsf{KBUlt}_{\mathrm{ne}} \to \mathsf{KBUlt}_{\mathrm{ne}}$ has a unique fixed point up to isomorphism.*

5 Domain equations: from O-categories to M-categories

As another illustration of M-categories, we present a general construction that gives for every O-category $\mathscr{C}$ (see below) a derived M-category $\mathscr{D}$. In addition, the construction gives for every locally continuous mixed-variance functor F on $\mathscr{C}$ a locally contractive mixed-variance functor G on $\mathscr{D}$ such that a fixed point of G (necessarily unique, by Theorem 3.1) is the same as a fixed point of F that furthermore satisfies a 'minimal invariance' condition [12]. Thus, generalized domain equations can be solved in M-categories.

The construction generalizes an earlier one [6] which is for the particular category of pointed cpos and strict, continuous functions (or full subcategories thereof) and only works for a restricted class of functors that does not include general function spaces.

Rank-ordered cpos [6], independently discovered under the name 'uniform cpos' [7], arise from a particular instance of an M-category obtained from this construction. The extra metric information in that category (as compared with the underlying O-category) is useful in realizability models [1, 4].

An *O-category* [17] is a category $\mathscr{C}$ where each hom-set $\mathscr{C}(A,B)$ is equipped with an ω-complete partial order, usually written $\sqsubseteq$, and where each composition function is continuous with respect to these orders. A functor $F : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ is *locally continuous* if each function on hom-sets that it induces is continuous.

Assume now that $\mathscr{C}$ is an O-category such that each hom-set $\mathscr{C}(A,B)$ contains a least element $\bot_{A,B}$ and such that the composition functions of $\mathscr{C}$ are strict: $f \circ \bot_{A,B} = \bot_{A,C} = \bot_{B,C} \circ g$ for all f and g. We construct an M-category $\mathscr{D}$ of 'rank-ordered $\mathscr{C}$-objects' as follows. An object $(A,(\pi_n)_{n\in\omega})$ of $\mathscr{D}$ is a pair consisting of an object A of $\mathscr{C}$ and a family of endomorphisms $\pi_n : A \to A$ in $\mathscr{C}$ that satisfies the following requirements:

(1) $\pi_0 = \bot_{A,A}$.

(2) $\pi_m \sqsubseteq \pi_n$ for all $m \leq n$.

(3) $\pi_m \circ \pi_n = \pi_n \circ \pi_m = \pi_{\min(m,n)}$ for all m and n.

(4) $\bigsqcup_{n\in\omega} \pi_n = id_A$.

Then, a morphism from $(A,(\pi_n)_{n\in\omega})$ to $(A',(\pi'_n)_{n\in\omega})$ in $\mathscr{D}$ is a morphism f from A to A' in $\mathscr{C}$ satisfying that $\pi'_n \circ f = f \circ \pi_n$ for all n. Composition and identities in $\mathscr{D}$ are the same as in $\mathscr{C}$. Finally, the distance function on a hom-set $\mathscr{D}((A,(\pi_n)_{n\in\omega}),(A',(\pi'_n)_{n\in\omega}))$ is defined as follows: $d(f,g) = 2^{-\max\{n\in\omega \mid \pi'_n\circ f = \pi'_n\circ g\}}$ if $f \neq g$, and $d(f,g) = 0$ otherwise. (One can show using conditions (1)-(4) above that this function is in fact well-defined.)

Proposition 5.1. *$\mathscr{D}$ is an M-category.*

Now let $F : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathscr{C}$ be a locally continuous functor. We construct a locally contractive functor $G : \mathscr{D}^{\mathrm{op}} \times \mathscr{D} \to \mathscr{D}$ from F. On objects, G is given by

$$G((A,(\pi_n^A)_{n\in\omega}),\,(B,(\pi_n^B)_{n\in\omega})) = (F(A,B),(\pi_n^{A,B})_{n\in\omega})$$

where $\pi_0^{A,B} = \bot$ and $\pi_{n+1}^{A,B} = F(\pi_n^A,\pi_n^B)$ for all n. On morphisms, G is the same as F, i.e., $G(f,g) = F(f,g)$. One can verify that G is well-defined and furthermore locally contractive with factor $1/2$.

Proposition 5.2. *Let A be an object of $\mathscr{C}$. The following two conditions are equivalent. (1) There exists an isomorphism $i : F(A,A) \to A$ such that $id_A = fix(\lambda e^{\mathscr{C}(A,A)}.\, i \circ F(e,e) \circ i^{-1})$. (Here fix is the least-fixed-point operator.) (2) There exists a family of morphisms $(\pi_n)_{n\in\omega}$ such that $\overline{A} = (A,(\pi_n)_{n\in\omega})$ is the unique fixed-point of G up to isomorphism.*

It remains to discuss how completeness properties of $\mathscr{C}$ transfer to $\mathscr{D}$. One can show that the forgetful functor from $\mathscr{D}$ to $\mathscr{C}$ creates terminal objects and limits of ω^{op}-chains of split epis. Alternatively, by imposing an additional requirement on $\mathscr{C}$ one can show that the forgetful functor creates *all* limits: for a given limit in $\mathscr{C}$, the induced bijection between cones and mediating morphisms must be an isomorphism in the category of cpos (where cones are ordered pointwise, using the order on each hom-set). That requirement is in particular satisfied by the usual concrete categories of cpos.

References

[1] M. Abadi and G. D. Plotkin. A per model of polymorphism and recursive types. In J. Mitchell, editor, *Proceedings of the Fifth Annual IEEE Symposium on Logic in Computer Science*, pages 355–365, Philadelphia, Pennsylvania, June 1990. IEEE Computer Society Press.

[2] F. Alessi, P. Baldan, and G. Bellè. A fixed-point theorem in a category of compact metric spaces. *Theoretical Computer Science*, 146(1&2):311–320, 1995.

[3] F. Alessi, P. Baldan, G. Bellè, and J. J. M. M. Rutten. Solutions of functorial and non-functorial metric domain equations. *Electronic Notes in Theoretical Computer Science*, 1:1–12, 1995.

[4] R. M. Amadio. Recursion over realizability structures. *Information and Computation*, 91(1):55–85, 1991.

[5] P. America and J. J. M. M. Rutten. Solving reflexive domain equations in a category of complete metric spaces. *Journal of Computer and System Sciences*, 39(3):343–375, 1989.

[6] C. Baier and M. E. Majster-Cederbaum. The connection between initial and unique solutions of domain equations in the partial order and metric approach. *Formal Aspects of Computing*, 9(4): 425–445, 1997.

[7] L. Birkedal, K. Støvring, and J. Thamsborg. Realizability semantics of parametric polymorphism, general references, and recursive types. In L. de Alfaro, editor, *Foundations of Software Science and Computational Structures, 12th International Conference, FOSSACS 2009*, number 5504 in Lecture Notes in Computer Science, pages 456–470, York, United Kingdom, Mar. 2009. Springer.

[8] L. Birkedal, K. Støvring, and J. Thamsborg. The category-theoretic solution of recursive metric-space equations, 2009. Manuscript, submitted to journal. Available from `http://itu.dk/people/kss/papers/metric-equations.pdf`.

[9] F. van Breugel and J. Warmerdam. Solving domain equations in a category of compact metric spaces. Report CS-R9424, CWI, Amsterdam, 1994.

[10] J. W. de Bakker and J. Zucker. Processes and the denotational semantics of concurrency. *Information and Control*, 54:70–120, 1982.

[11] M. Kostanek and P. Waszkiewicz. On the influence of domain theory on Q-categories, 2009. Manuscript, submitted. Available from `http://tcs.uj.edu.pl/Waszkiewicz/`.

[12] A. M. Pitts. Relational properties of domains. *Information and Computation*, 127:66–90, 1996.

[13] J. J. M. M. Rutten. Elements of generalized ultrametric domain theory. *Theoretical Computer Science*, 170(1-2):349–381, 1996.

[14] J. J. M. M. Rutten and D. Turi. On the foundations of final semantics: Non-standard sets, metric spaces, partial orders. Report CS-R9241, CWI, Amsterdam, 1992.

[15] J. Schwinghammer, L. Birkedal, B. Reus, and H. Yang. Nested Hoare triples and frame rules for higher-order store. In *Proceedings of the 18th EACSL Annual Conference on Computer Science Logic*, 2009. To appear.

[16] M. B. Smyth. Topology. In S. Abramsky, D. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*. Oxford University Press, 1992.

[17] M. B. Smyth and G. D. Plotkin. The category-theoretic solution of recursive domain equations. *SIAM Journal on Computing*, 11(4):761–783, 1982.

[18] K. R. Wagner. *Solving Recursive Domain Equations with Enriched Categories*. PhD thesis, Carnegie Mellon University, Pittsburgh, Pennsylvania, 1994.