Exact Threshold Circuits

Kristoffer Arnsfelt Hansen
Aarhus University
Århus, Denmark
Email: arnsfelt@cs.au.dk

Vladimir V. Podolskii
Steklov Mathematical Institute
Moscow, Russia
Email: podolskii@mi.ras.ru

***Abstract*—We initiate a systematic study of constant depth Boolean circuits built using exact threshold gates. We consider both unweighted and weighted exact threshold gates and introduce corresponding circuit classes. We next show that this gives a hierarchy of classes that seamlessly interleave with the well-studied corresponding hierarchies defined using ordinary threshold gates.**

A major open problem in Boolean circuit complexity is to provide an explicit super-polynomial lower bound for depth two threshold circuits. We identify the class of depth two exact threshold circuits as a natural subclass of these where also no explicit lower bounds are known. Many of our results can be seen as evidence that this class is a strict subclass of depth two threshold circuits — thus we argue that efforts in proving lower bounds should be directed towards this class.

***Keywords*-Boolean Circuits; Threshold Functions; Exact Threshold Functions;**

I. Introduction

Linear threshold functions are Boolean functions defined by an intersection of a halfspace with the Boolean n-cube. Their importance have since long been established in many fields of computer science, cf. [24], [31], [29]. The study of Boolean *circuits* built from threshold functions, or gates, was initiated by Parberry and Schnitger [32]. They considered the class of constant depth circuits built from unweighted threshold gates, i.e. majority gates — the class that has since been named TC^0. One may in general consider constant depth circuits built from weighted threshold functions. From the work of Chandra, Stockmeyer and Vishkin [9] and Pippenger [33] it follows that any threshold function can be computed by polynomial size TC^0 circuits. Thus if one disregards the exact depth of the circuits, one may freely use weighted threshold gates to define TC^0 circuits.

The seminal work of Hajnal *et al.* [20] provided the first methods for analyzing the computational limitations of threshold circuits. Their "ϵ–discriminator" lemma reduces the task of proving a size lower bound for computing a function f with a circuit consisting of a majority vote of subcircuits $C_1, \ldots, C_S$ to a question about correlation of any subcircuit with the function f. Using this together with Lindsey's lemma [11], [2] they showed that depth two circuits with MAJ gates must use at least $2^{(1/2-\epsilon)n}$ gates to compute the *inner product modulo 2* (IP_2) function on $2n$ variables. When weights are allowed in the bottom layer they show that at least $2^{(1/3-\epsilon)n}$ gates are required.

Since this work, much research have been directed towards proving strong lower bounds for threshold circuits. The vast part of this effort can be divided in the following two categories:

1) Study of subclasses of depth 2 circuits with a *weighted* threshold gate at the output.
2) Study of subclasses of depth 3 circuits with an *unweighted* threshold gate at the output.

By results of Goldmann, Håstad and Razborov [15] it is known that the latter category includes the former, and thus proving strong lower bounds for depth 3 majority circuits is the greater challenge. Even more striking, by results of Yao [38] and Beigel and Tarui [4] this class of circuits can in quasi–polynomial size simulate all of ACC^0, where ACC^0 is the class of polynomial size constant depth circuits built from AND, OR and MOD_m gates. In fact from these results it follows one can even simulate the class of $\mathsf{MAJ} \circ \mathsf{MAJ} \circ \mathsf{ACC}^0$ circuits, meaning circuits with two layers of majority gates taking ACC^0 circuits as input, by depth 3 majority circuits of quasi–polynomial size.

Most of the lower bounds for subclasses of depth 3 majority circuits employ the use of the ϵ–discriminator lemma, reducing the problem to a question about depth two majority circuits. For several such subclasses have these questions been answered successfully using probabilistic communication complexity. Håstad and Goldmann [23] showed that if the fanin of either the bottom or the middle layer is sufficiently limited then exponential lower bounds can be proved by multi–party or two–party communication complexity, respectively.

Not surprisingly, the connection to the class ACC^0 have continued to provide a constant supply of challenges to current research, e.g. [19], [6], [18], [10]. Nevertheless it is still the case that no strong lower bounds are known for most variants of depth three $\mathrm{MAJ} \circ \mathsf{ACC}^0$ circuits, leaving ample opportunities for further research.

We next turn to the smaller category of depth two threshold circuits with a weighted threshold gate at the output. Bruck [7] gave exponential lower bounds for polynomial threshold functions, which correspond to $\mathsf{THR} \circ \mathsf{MOD}_2$ circuits, a threshold of parity functions. Krause and Pudlak

[26] showed exponential lower bounds for $\mathsf{THR} \circ \mathsf{MOD}_m$ circuits in general. Goldmann [14] gave exponential lower bounds for constant depth AND/OR circuits with a threshold gate at the output, i.e. $\mathsf{THR} \circ \mathsf{AC}^0$ circuits. In a breakthrough result Forster proved a strong lower bound on the two–party unbounded error probabilistic communication complexity [12]. This enabled exponential circuit lower bounds for $\mathsf{THR} \circ \mathsf{MAJ}$ circuits [13] — this subclass include all the subclasses of $\mathsf{THR} \circ \mathsf{THR}$ where a lower bound had already been proved. In fact, it includes all subclasses that have previously been studied in the literature, thereby leaving only the question of strong lower bounds for the full class itself unanswered. In this work we will consider a new subclass, where no strong lower bounds are known, namely the class of depth two *exact threshold* circuits. Where a linear threshold function is defined by an intersection of the Boolean n–cube with a *halfspace*, linear exact threshold functions are Boolean functions defined by an intersection of the Boolean n-cube with a *hyperplane*.

Previously Roychowdhury, Orlitsky, and Siu [35] had pointed out that no lower bounds are known for depth two threshold circuits even if it is assumed that the linear function defining the output gate is always either 0 or 1. One can think of this special case as a *promise* circuit class. The class of depth two exact threshold circuit on the other hand is a class of circuits defined in the standard way, composed of Boolean gates — we remark that it also includes the promise class mentioned by Roychowdhury, Orlitsky, and Siu. Previous work on circuits with exact threshold functions is sparse [5], [17], [21], [22], and some only consider unweighted exact threshold functions.

In order to gain understanding of depth two exact threshold circuits we initiate a systematic study of circuit classes built using exact threshold functions in general. We consider two hierarchies of exact threshold circuits. One is formed by the classes of depth d polynomial size weighted exact threshold circuits for all constant d, and the other is the similar hierarchy formed by unweighted circuits. For the analogous hierarchies given by usual threshold gates it is known that they can be merged: depth d weighted threshold circuit of polynomial size can be simulated by depth $d+1$ unweighted threshold circuit of polynomial size [15]. We prove that the same is true for the case of exact threshold circuits. Moreover, we show that the resulting hierarchy for exact threshold circuits seamlessly interleave with the corresponding hierarchies defined using threshold gates. More precisely, we show that the class of depth d unweighted threshold circuits of polynomials size contains the class of depth d unweighted exact threshold circuits of polynomial size and is contained in the class of depth $d+1$ unweighted exact threshold circuits of polynomial size. We further prove that the same is true for weighted circuits. Finally we show separations between low depth circuits in these hierarchies and between other relevant low depth classes. It appears that the smallest class for which we do not know explicit lower bounds is the class of depth 2 weighted exact threshold circuits.

Most of our results are obtained using techniques developed for threshold circuits. In several cases we find that the perspective from exact threshold functions provide an illuminating perspective on these.

The rest of the paper is organized as follows. In Section II we define the Boolean functions and circuit classes we consider as well as provide some basic properties of these. In Section III we show inclusions between the newly defined classes and threshold circuit classes. To complement these results, in Section IV we derive separation between most of the circuit classes, for which we are able to prove lower bounds. In Section V we consider one more class and show its position among other classes.

II. Preliminaries

A. Boolean functions

We consider here a Boolean function f to be a function $f : \{0,1\}^n \to \{0,1\}$. As is usual we will in fact always have a family of such functions in mind, one for each input length n. Our main focus will be threshold style functions, defined by linear equations and inequalities of Boolean variables. Let $x_1, \ldots, x_n \in \{0,1\}$ be Boolean variables. Let $w_1, \ldots, w_n$ and t be real numbers. An *exact threshold* function is a Boolean function that decides if a linear equation of the following form holds:

$$w_1 x_1 + \cdots + w_n x_n = t \ .$$

Similarly, a *threshold* function is a Boolean function that decides if a linear inequality of the following form holds:

$$w_1 x_1 + \cdots + w_n x_n \geq t \ .$$

More precisely, for $w = (w_1, \ldots, w_n) \in \mathbf{R}^n$ and $t \in \mathbf{R}$ we define the function $\mathrm{ETHR}_{w,t}$ by $\mathrm{ETHR}_{w,t}(x) = 1$ if and only if $\sum_{i=1}^n w_i x_i = t$ and the function $\mathrm{THR}_{w,t}$ by $\mathrm{THR}_{w,t}(x) = 1$ if and only if $\sum_{i=1}^n w_i x_i \geq t$.

We call $w_1, \ldots, w_n$ the *weights* and t the *threshold*. We say that these weights and threshold are a *realization* of the Boolean function they define. Note that exact threshold and threshold functions have many different realizations. One may observe that without loss of generality one can assume the real valued weights and the real valued threshold are integers. In fact one may assume that the weights are integers of absolute size at most $2^{O(n \log n)}$ [30], [3].

Proposition 1 (Muroga *et al.*;Babai *et al.*)**.**

1) *Any threshold function on n variables is realized by integer weights of absolute size at most $(n+1)^{(n+1)/2}/2^n$.*
2) *Any exact threshold function on n variables is realized by integer weights of absolute size at most $n^{n/2+1}$.*

We shall single out the special case when all weights are 1 and the threshold is $n/2$. We define the function EMAJ by $\mathrm{EMAJ}(x) = 1$ if and only if $\sum_{i=1}^{n} x_i = n/2$ and the function MAJ by $\mathrm{MAJ}(x) = 1$ if and only if $\sum_{i=1}^{n} x_i \geq n/2$.

We shall require other Boolean functions as well. These include the unary NOT function as well as the usual AND, OR, and XOR functions of n Boolean variables. We shall denote the number of inputs to these by a subscript, e.g. by AND_k we denote the Boolean AND function of k Boolean variables. In addition to these we also consider arbitrary symmetric Boolean functions, i.e. Boolean functions whose value only depend on the number of inputs that are 1.

It will be useful to have a notation for the different classes of functions we consider. Let ETHR and THR denote the class of $\mathrm{ETHR}_{w,t}$ and $\mathrm{THR}_{w,t}$ functions for all w and t. Let EMAJ, MAJ, AND, OR and XOR denote the class of all EMAJ, MAJ, AND, OR and XOR functions. Let SYM denote the class of all symmetric Boolean functions.

We will also use types of *promise* gate. We denote by LIN a linear combination of inputs, with coefficients of polynomially large absolute value, satisfying the promise that the linear combination is always either 0 or 1. A special case of this is a *disjoint* OR, by which we mean a disjunction satisfying that at most one of the inputs are satisfied at the same time.

All the functions defined so far will be used as primitives in defining circuit classes. For separating different classes of circuit we will consider several other functions as well. We shall recall the definition of some of these below – others will be defined as needed. Let $x, y \in \{0,1\}^n$. The *greater than* function GT is defined by $\mathrm{GT}(x,y) = 1$ if and only if $x \geq y$, where the comparison is between x and y considered as binary representation of integers. That is, $\mathrm{GT}(x,y) = 1$ if and only if $\sum_{i=1}^{n}(x_i - y_i)2^{i-1} \geq 0$, which shows that GT is a threshold function. The (sequence) equality function EQ is defined by $\mathrm{EQ}(x,y) = 1$ if and only if for all i, $x_i = y_i$. We thus have $\mathrm{EQ}(x,y) = 1$ if and only if $\sum_{i=1}^{n}(x_i - y_i)2^{i-1} = 0$, which shows that EQ is an exact threshold function. The *disjointness* function DISJ is defined by $\mathrm{DISJ}(x,y) = 1$ if and only if for all i, $(x_i = 0 \vee y_i = 0)$. That is, if we consider x and y as characteristic vectors of subsets of $\{1,\ldots,n\}$ then $\mathrm{DISJ}(x,y) = 1$ if and only if $x \cap y = \emptyset$. We shall denote the negations of GT, EQ and DISJ as $\overline{\mathrm{GT}}$, $\overline{\mathrm{EQ}}$ and $\overline{\mathrm{DISJ}}$, respectively.

B. Circuit classes

We consider unbounded fanin Boolean circuits built from the families of Boolean functions we defined. We shall assume familiarity with the basic notions of circuits. Inputs to the circuits are allowed to be Boolean variables and their negations as well as the Boolean constants 0 and 1. As with Boolean functions we will in fact always have a family of Boolean circuits in mind, one for every number of inputs. By the *size* of a circuit we shall refer to the number of *wires* rather than number of *gates*. The *depth* of a circuit is the length of the longest path from an input of the circuit to the output gate of the circuit.

We shall now define the main classes we consider. Let $i \geq 1$. Then ELT_i is the class of depth i polynomial size circuits built using ETHR gates. Similarly, LT_i is the class of depth i polynomial size circuits built using THR gates. We also define "small-weights" versions of these. Define $\widehat{\mathsf{ELT}}_i$ and $\widehat{\mathsf{LT}}_i$ to be the subclasses of ELT_i and LT_i where the weights of all gates are restricted to be integers of polynomially large absolute value.

We define TC^0 to be the class of circuits computed by constant depth polynomial size circuits built entirely using MAJ gates. We refine this to classes of specific depths, by letting TC^0_i denote the subclass of depth i, for $i \geq 1$. Thus $\mathsf{TC}^0 = \cup_{i=1}^{\infty}\mathsf{TC}^0_i$. Observing that weights of polynomial size may be simulated by duplication of wires to an unweighted gate and negative weights may be simulated by an additional negation (which can afterwards be moved down to the input level) we have the following simple fact:

Proposition 2. *For all $i \geq 1$ we have, $\widehat{\mathsf{LT}}_i = \mathsf{TC}^0_i$.*

For classes of circuits $\mathcal{C}_1$ and $\mathcal{C}_2$ let $\mathcal{C}_1 \circ \mathcal{C}_2$ denote the class of polynomial size circuits that consists of a circuit from $\mathcal{C}_1$ that is fed as inputs the outputs of circuits from $\mathcal{C}_2$. With this definition we have the following fact similar to Proposition 2. We note however that the proof of the second statement of part 2 is not as simple, since handling negations requires to change the structure of the circuit. This can be done using the methods of Theorem 7.

Proposition 3. *For any $i \geq 1$ we have*

1) a) $\mathsf{LT}_i = \overbrace{\mathsf{THR} \circ \cdots \circ \mathsf{THR}}^{i \text{ times}}$.

   b) $\widehat{\mathsf{LT}}_i = \overbrace{\mathsf{MAJ} \circ \cdots \circ \mathsf{MAJ}}^{i \text{ times}}$.

2) a) $\mathsf{ELT}_i = \overbrace{\mathsf{ETHR} \circ \cdots \circ \mathsf{ETHR}}^{i \text{ times}}$.

   b) $\widehat{\mathsf{ELT}}_i = \overbrace{\mathsf{EMAJ} \circ \cdots \circ \mathsf{EMAJ}}^{i \text{ times}}$.

We define AC^0 to be the class of circuits computed by constant depth polynomial size circuits built from AND and OR gates. We remark that by De Morgan's laws, AC^0 could also be defined as circuits built entirely from AND and NOT gates, or entirely from OR and NOT gates.

C. Basic Properties

As previously mentioned, from the work of Chandra, Stockmeyer and Vishkin [9] and Pippenger [33] it follows that any threshold function can be computed by TC^0 circuits, and as a consequence we also have $\mathsf{TC}^0 = \cup_{i=1}^{\infty}\mathsf{LT}_i$. Sui and Bruck first considered the question of a depth efficient simulation and proved that any threshold function can be

computed by $\widehat{\mathsf{LT}}_3$ circuits [37]. Goldmann, Håstad and Razborov proved that only $\widehat{\mathsf{LT}}_2$ circuits are required [15]. In fact they obtained the following stronger statement:

Theorem 4 (Goldmann, Håstad and Razborov)**.** *For all $i \geq 1$ we have $\mathsf{MAJ} \circ \mathsf{LT}_i = \mathsf{MAJ} \circ \widehat{\mathsf{LT}}_i$. In particular $\mathsf{MAJ} \circ \mathsf{THR} = \mathsf{MAJ} \circ \mathsf{MAJ}$.*

These simulation results have since been simplified a number of times [16], [25], [1]. Next we show a simple connection between ETHR gates and THR gates.

Proposition 5.

1) $\mathsf{ETHR} \subseteq \mathsf{THR} \circ \mathsf{AND}_2$.
2) $\mathsf{EMAJ} \subseteq \mathsf{MAJ} \circ \mathsf{AND}_2$.

Proof: Suppose that an exact threshold function is given by $L(x) = 0$, where

$$L(x) = w_1 x_1 + \cdots + w_n x_n - t \ .$$

Then we have $L(x) = 0$ if and only if $(L(x))^2 \leq 0$. For each degree 2 term in the polynomial $(L(x))^2$ we have an AND gate of the variables. We feed the outputs of these as well as variables corresponding to terms of degree 1 to a threshold gate using as weights the coefficients of the terms. If the coefficients of $L(x)$ are polynomially bounded then the coefficients of $(L(x))^2$ are polynomially bounded as well. ∎

Proposition 6. *Classes with exact threshold gates satisfy the following closure under* AND *properties:*

1) $\mathsf{AND}_k \circ \mathsf{EMAJ} = \mathsf{EMAJ}$*, for any positive integer k.*
2) $\mathsf{AND} \circ \mathsf{ETHR} = \mathsf{ETHR}$.
3) $\mathsf{AND} \circ \mathsf{EMAJ} \subseteq \mathsf{EMAJ} \circ \mathsf{AND}_2$.

Proof: For the first two statements, we consider the AND of k exact threshold gates. Suppose that the jth exact threshold function corresponds to the linear equation $L_j(x) = 0$. Let B be the smallest integer such that $|L_j(x)| \leq B$ for all $x \in \{0,1\}^n$ and for all j. We then have for all $x \in \{0,1\}^n$, that $\sum_{j=1}^k B^j L_j(x) = 0$ if and only if $L_j(x) = 0$ for all j. It is easy to see that the weights of this new exact threshold function are bounded by B^{k+1}. In case B is polynomially bounded and k is constant this is polynomially bounded as well.

Finally for the last statement, suppose that k exact majority functions are given by linear functions $L_1, \ldots, L_k$. We then have that $L_j(x) = 0$ holds for all j if and only if it holds that $\sum_{j=1}^k (L_j(x))^2 = 0$. We may evaluate this by an $\mathsf{EMAJ} \circ \mathsf{AND}_2$ circuit as in Proposition 5. ∎

III. Circuit class inclusions

Theorem 7. *Any* SYM *gate is a disjoint OR of* EMAJ *gates. Any* THR *gate is a disjoint OR of* ETHR *gates. Thus we have*

1) $\mathsf{LIN} \circ \mathsf{MAJ} = \mathsf{LIN} \circ \mathsf{SYM} = \mathsf{LIN} \circ \mathsf{EMAJ}$.
2) $\mathsf{LIN} \circ \mathsf{THR} = \mathsf{LIN} \circ \mathsf{ETHR}$.

Proof: 1. Hajnal *et al.* essentially proved $\mathsf{LIN} \circ \mathsf{MAJ} = \mathsf{LIN} \circ \mathsf{SYM}$ [20]. It remains to prove that any symmetric function f on n variables is a disjoint OR of EMAJ gates. The function f is given by a set $S \subseteq \{0, 1, \ldots, n\}$, where $f(x) = 1$ if and only if $\sum_{i=1}^n x_i \in S$. We can thus write this as a disjoint OR of the EMAJ gates given by $\sum_{i=1}^n x_i = t$ for all $t \in S$.

2. The inclusion $\mathsf{ETHR} \subseteq \mathsf{LIN} \circ \mathsf{THR}$ is simple since $\sum_{i=1}^n w_i x_i = t$ is the difference of $(\sum_{i=1}^n w_i x_i \geq t)$ and $(\sum_{i=1}^n w_i x_i \geq t + \epsilon)$, where $\epsilon > 0$ is sufficiently small. We next show that any THR gate is a disjoint OR of ETHR gates. The proof uses insight from [25], [1]. Suppose we have threshold gate $\sum_{i=1}^n w_i x_i + w_0 \geq 0$, defined by $F(x) = \sum_{i=1}^n w_i x_i + w_0$. Let L be the minimal integer such that $|w_i| < 2^L$ for all i. By the first part of Proposition 1 we may assume that $L = O(n \log n)$. Let us make the following definitions (almost as in [1]) for all $l \leq L$:

$$w_i^{(l)} = \lfloor w_i / 2^l \rfloor$$
$$F^{(l)}(x) = \sum_{i=1}^n w_i^{(l)} x_i + w_0^{(l)}$$
$$E^{(l)}(x) = F^{(l-1)}(x) - 2F^{(l)}(x)$$

Note that $F^{(l)}(x) \leq F^{(l-1)}(x)$ and moreover $E^{(l)}(x) \geq 0$. Also note that $E^{(l)}(x) \leq n + 1$. Let

$$E_{max} = \max_l \max_{x \in \{0,1\}^n} E^{(l)}(x).$$

Now we claim that $F(x) \geq 0$ if and only if

$$\bigvee_{l=0}^{L} \Big(F^{(l)}(x) \in [0, E_{max}] \wedge F^{(l-1)}(x) \in [E_{max} + 1, 3E_{max}] \Big) \quad (1)$$

and moreover that the OR gate in the formula above is disjoint. For $l = 0$ we mean that the right part of the conjunction in (1) is true. The proof of the equivalence follows from the claim below.

Claim 1.

1) If $F^{(l-1)}(x) > 3E_{max}$ then $F^{(l)}(x) > E_{max}$.
2) If $F^{(l-1)}(x) > E_{max}$ then $F^{(l)}(x) > 0$.
3) If $F(x) < 0$ then $F^{(l)}(x) < 0$ for all l.
4) $F^{(L)}(x) \leq 0$

The first part of the claim follows from the following calculation:

$$F^{(l)}(x) = \frac{F^{(l-1)}(x) - E^{(l-1)}(x)}{2} > \frac{3E_{max} - E_{max}}{2} = E_{max} \ .$$

The proof of the second part of the claim is completely analogous. The third part follows from the facts that $F(x) \equiv$

$F^{(0)}(x)$ and $F^{(l)}(x) \leq F^{(l-1)}(x)$. The last part of the claim is obvious from the definition of $w_i^{(L)}$. Now we are in position to complete the proof. It is obvious how to write $F^{(l)}(x) \in [a,b]$ as a disjoint OR of ETHR gates. Furthermore, a conjunction of such two distributes to a disjoint OR of ETHR gates, using Proposition 6. Combining everything yields a disjoint OR of ETHR gates as well. ■

Corollary 8. *Let $\mathcal{C}$ denote one of the classes* LIN, MAJ, THR, EMAJ, ETHR. *Then for all $i \geq 1$ we have $\mathcal{C} \circ \widehat{\mathsf{LT}}_i = \mathcal{C} \circ \widehat{\mathsf{ELT}}_i$ and $\mathcal{C} \circ \mathsf{LT}_i = \mathcal{C} \circ \mathsf{ELT}_i$.*

Proof: To prove this corollary we apply Theorem 7 successively to all layers of the circuit using the fact that a LIN-gate is "contained" in MAJ, THR, EMAJ, ETHR gates. ■

The above proof of the second part of Theorem 7 used insight from the proof due to Hofmeister of the result that $\mathsf{THR} \subseteq \mathsf{MAJ} \circ \mathsf{MAJ}$ [25]. His proof was logically in two parts, yet no clear statement resulted from the first part. We believe that the viewpoint of exact threshold functions leads to a conceptually even simpler proof of this important result, even though additional arguments were needed above. Nothing is lost from doing this, however — we may now carry out that the second part of his proof in a simpler way. We sketch how to do this below.

Sketch of proof: By Theorem 7 any THR function is a disjoint OR of polynomially many ETHR gates. We will now use the technique of "Chinese remaindering" as in the proof of Theorem 10, but with more distinct primes. We will ensure that in case the ETHR gate is not 1, the corresponding equation will only hold modulo a polynomially small fraction of the primes. Suppose that we have m ETHR gates, k primes and ensure that either an equation holds modulo all k primes or for at most h primes. We now sum all outputs of the corresponding EMAJ gates. In case the THR gate is 1, the number of EMAJ gates that evaluate to 1 is between k and $k + hm$. In case the THR gate is 0, the number of EMAJ gates that evaluate to 1 is at most hm. Thus as long as we choose the parameters such that $hm < k$ we may distinguish this by a MAJ gate. ■

We can reformulate the result of Theorem 7 in the following interesting statement: Any intersection of the Boolean cube with a halfspace can be partitioned into polynomially many disjoint sets such that each set is the intersection of the Boolean cube with a hyperplane.

As shown, in the case of polynomially bounded weights one can choose these sets such that they correspond to parallel hyperplanes. One may wonder if this is true in general. We show that for the GT function exponentially many such hyperplanes are required.

Proposition 9. *Suppose that $w_1, \ldots, w_n$ and $w'_1, \ldots, w'_n$ are weights and $t_1, \ldots, t_k$ are a set of thresholds such that $\mathrm{GT}(x,y) = 1$ if and only if there exists j such that $\sum_{i=1}^n w_i x_i + w'_i y_i = t_j$. Then k must be at least 2^n.*

Proof: First observe that all sums of the form $\sum_{i=1}^n w'_i y_i$ must be distinct. Now fix $x_i = 1$ for all i. Then it is the case that $\mathrm{GT}(x,y) = 1$ for all y. Suppose for contradiction that y and y' are distinct but $\mathrm{GT}(x,y)$ and $\mathrm{GT}(x,y')$ are certified by the same t_j. Then we have $\sum_{i=1}^n w'_i y_i = \sum_{i=1}^n w'_i y'_i$. Assume $\mathrm{GT}(y,y') = 1$, say. Then we have $\mathrm{GT}(y',y') = 1$ and $\mathrm{GT}(y',y) = 0$. From the first we have there exists t_l such that $\sum_{i=1}^n w_i y'_i + w'_i y'_i = t_l$, but then we also have $\sum_{i=1}^n w_i y'_i + w'_i y_i = t_l$ which would mean $\mathrm{GT}(y',y) = 1$. ■

Theorem 10. *Any function computed by an exact threshold gate is computed by a depth 2 small weight exact threshold circuit, that is,* $\mathsf{ETHR} \subseteq \mathsf{EMAJ} \circ \mathsf{EMAJ}$.

Proof: First we prove that $\mathsf{ETHR} \subseteq \mathsf{AND} \circ \mathsf{SYM}$. For this we will use the standard technique of "Chinese remaindering". Consider an exact threshold function given by $w_1 x_1 + \cdots + w_n x_n = t$. Define the function E by $E(x) = \sum_{i=1}^n w_i x_i - t$. Let W be an integer such that $|E(x)| < W$ for all $x \in \{0,1\}^n$. By Proposition 1 we may choose $W = 2^{O(n \log n)}$. Let $p_1, \ldots, p_k$ be the k smallest primes such that $p_1 \cdots p_k \geq W$. We then have $p_k = O(n \log n)$, by the prime number theorem. In order to compute whether $E(x) = 0$, by the Chinese remainder theorem we may instead check whether $E(x) \equiv 0 \pmod{p_j}$ for all j. Now consider a fixed j. Let E_j be the function obtained from E by reducing all coefficients p_j, that is define the function E_j by $E_j(x) = \sum_{i=1}^n (w_i \mod p_j) x_i + (-t \mod p_j)$. Now we have $E(x) \equiv 0 \pmod{p_j}$ if and only if $E_j(x) \in \{0, p_j, \ldots, np_j\}$. The latter may be checked by a single SYM gate that takes $(w_i \mod p_j)$ copies of x_i as input for all i. Taking the AND function of all these gives an $\mathsf{AND} \circ \mathsf{SYM}$ circuit computing the given exact threshold function. Now we can derive the stated result. Using Theorem 7 we have $\mathsf{AND} \circ \mathsf{SYM} \subseteq \mathsf{EMAJ} \circ \mathsf{SYM} = \mathsf{EMAJ} \circ \mathsf{EMAJ}$. ■

We are now in position to show that the exact threshold classes form a hierarchy interleaving with the hierarchy of threshold classes, see Figure 2 and Figure 1.

Theorem 11. *For all $i \geq 1$ we have*

1) $\widehat{\mathsf{ELT}}_i \subseteq \mathsf{ELT}_i \subseteq \widehat{\mathsf{ELT}}_{i+1}$. *(This is an analog of Theorem 4 for exact-threshold circuits)*
2) $\widehat{\mathsf{LT}}_i \subseteq \widehat{\mathsf{ELT}}_{i+1} \subseteq \widehat{\mathsf{LT}}_{i+1}$.
3) $\mathsf{LT}_i \subseteq \mathsf{ELT}_{i+1} \subseteq \mathsf{LT}_{i+1}$.

Proof:

1) The inclusion $\widehat{\mathsf{ELT}}_i \subseteq \mathsf{ELT}_i$ is obvious. Note that the inclusion $\mathsf{ELT}_i \subseteq \widehat{\mathsf{ELT}}_{i+1}$ was proved in Theorem 10

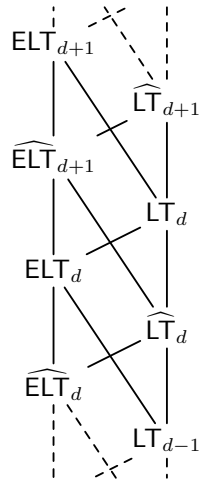

Figure 1. Upper levels of the hierarchy.

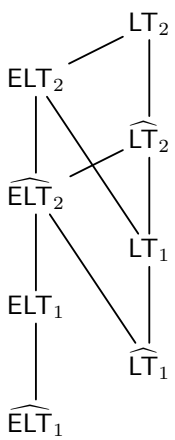

Figure 2. First two levels of the hierarchy.

for the case $i = 1$. For $i > 1$ we have

$$\begin{aligned}\mathsf{ELT}_i \subseteq\ & \mathsf{EMAJ} \circ \mathsf{EMAJ} \circ \mathsf{ELT}_{i-1} \subseteq \\ & \mathsf{EMAJ} \circ \mathsf{MAJ} \circ \mathsf{LT}_{i-1} \subseteq \\ & \mathsf{EMAJ} \circ \mathsf{MAJ} \circ \widehat{\mathsf{LT}}_{i-1} \subseteq \\ & \mathsf{EMAJ} \circ \mathsf{EMAJ} \circ \widehat{\mathsf{ELT}}_{i-1} = \widehat{\mathsf{ELT}}_{i+1}.\end{aligned}$$

We apply here Theorem 10, Theorem 7, Theorem 4 and Corollary 8.

2) First, we have $\widehat{\mathsf{LT}}_i \subseteq \mathsf{EMAJ} \circ \widehat{\mathsf{LT}}_i = \widehat{\mathsf{ELT}}_{i+1}$. For the proof of the second inclusion we apply Propositions 5, 6 and Corollary 8:

$$\begin{aligned}\widehat{\mathsf{ELT}}_{i+1} =\ & \mathsf{EMAJ} \circ \widehat{\mathsf{ELT}}_i \subseteq \\ & \mathsf{MAJ} \circ \mathsf{AND}_2 \circ \mathsf{EMAJ} \circ \widehat{\mathsf{ELT}}_{i-1} = \\ & \mathsf{MAJ} \circ \mathsf{EMAJ} \circ \widehat{\mathsf{ELT}}_{i-1} = \\ & \mathsf{MAJ} \circ \widehat{\mathsf{ELT}}_i = \widehat{\mathsf{LT}}_{i+1}.\end{aligned}$$

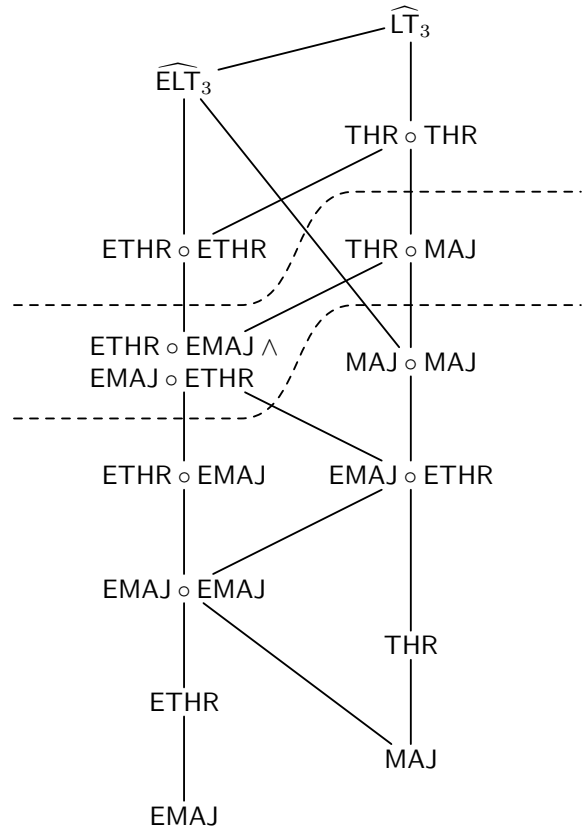

Figure 3. The entire low–level hierarchy. For classes below the top–most dashed line strong size lower bounds are known. For classes below the bottom–most dashed line separations are known to all other classes.

3) The proof of these inclusions is analogous to the previous part of the theorem.

■

From now on we shall consider only circuits of low depth and study relations between them. The classes we choose to consider can be found in Figure 3. We denote the class $(\mathsf{ETHR}\circ\mathsf{EMAJ})\wedge(\mathsf{EMAJ}\circ\mathsf{ETHR})$ by AETM in the text and we will consider it separately in Section V. All inclusions on the Figure 3 not concerning AETM are either already proved or are obvious. Now we are going to prove separations between most of these classes.

IV. Circuit class separations

A. Known and simple separations

We start from the bottom of the Figure 3. It is easy to see that the function MAJ_2 is not in ETHR (it should be one on inputs $(0,1)$, $(1,0)$, $(1,1)$ but they are not on the same hyperplane). On the other hand note that the function EQ_2 is the negation of XOR_2. And it is known that this function is not in THR (see [28]). These two functions separate EMAJ, ETHR, MAJ, THR from all other classes. A function separating $\mathsf{THR} \circ \mathsf{MAJ}$ from $\mathsf{MAJ} \circ \mathsf{MAJ}$ was constructed in [15]. In Section IV-D we shall explain that

their proof in fact gives us much more. We do not know whether the class $\mathsf{THR} \circ \mathsf{THR}$ differs from $\mathsf{THR} \circ \mathsf{MAJ}$.

B. Rank based lower bounds

Krause and Waack developed a rank based technique for proving lower bounds for $\mathsf{MOD}_m \circ \mathsf{SYM}$ and $\mathsf{MAJ} \circ \mathsf{SYM}$ circuits [27], the *variation rank* method. We will adapt this method to prove lower bounds for $\mathsf{ETHR} \circ \mathsf{SYM}$ circuits. We consider functions and circuits of two sets of Boolean inputs x and y, $x, y \in \{0,1\}^n$. To such a Boolean function f we associate an $2^n \times 2^n$ matrix M_f, the "communication matrix", by letting entry (x, y) be $f(x, y)$. We say that two $2^n \times 2^n$ matrices A and B are equality–equivalent, if for all x and y it holds that $A_{xy} = 0$ if and only if $B_{xy} = 0$. We define the equality rank[1] of A to be the minimum rank of any real–valued matrix B that is equality–equivalent to A.

Proposition 12 (Krause and Waack)**.** *Suppose a Boolean function f in variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ is computed by a circuit of size S consisting of a single* SYM *gate. Then the matrix M_f either has at most $S/2 + 1$ distinct nonzero rows or at most $S/2 + 1$ distinct nonzero columns. Hence the rank of M_f is at most $S/2 + 1$.*

Proposition 13. *Suppose a Boolean function f in variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ is computed by a* $\mathsf{ETHR} \circ \mathsf{SYM}$ *circuit C of size S. Then the equality rank of the communication matrix of the negation of f, $M_{\neg f}$, is less than S.*

Proof: Let $C_1, \ldots, C_k$ be the subcircuits of C that consists of single SYM gates, and assume that C_1 is of size S_i. Thus $S = k + \sum_{i=1}^{k} S_i$. Let $w_1, \ldots, w_k$ be the weights of the output gate and t the threshold. We then have that $f(x, y) = 1$ if and only if $\sum_{i=1}^{k} w_i C_i(x, y) = t$. Thus the following matrix is is equality–equivalent to $M_{\neg f}$:

$$\left(\sum_{i=1}^{k} w_i M_{C_i(x,y)} \right) - tJ \ .$$

Here J is the $2^n \times 2^n$ matrix with all entries being 1. We can thus conclude:

$$\begin{aligned} &\operatorname{rank}\left(\left(\sum_{i=1}^{k} w_i M_{C_i(x,y)} \right) - tJ \right) \leq \\ &1 + \sum_{i=1}^{k} \operatorname{rank}(M_{C_i(x,y)}) \leq \\ &1 + \sum_{i=1}^{k} (S_i/2 + 1) = \\ &1 + (k + S)/2 < S \ . \end{aligned}$$

■

[1]This notion is also known as minimal structural rank in the control theory literature. We prefer the term equality rank here as an analogous notion to sign rank.

Proposition 14. *The equality–rank of each of M_{EQ_n}, M_{GT_n} and M_{DISJ_n} is 2^n. The equality rank of $M_{\overline{\mathrm{GT}}_n}$ is $2^n - 1$.*

Proof: First we observe that the first three matrices are triangular $2^n \times 2^n$ matrices where all entries on the main diagonal are 1, and the fourth matrix contains such a matrix of dimension $2^n - 1 \times 2^n - 1$. In the case of M_{EQ_n} we simply have the identity matrix and in case of M_{GT_n} we have a lower triangular matrix where all entries on and below the main diagonal are 1. For $M_{\overline{\mathrm{GT}}_n}$, by deleting the first row and first column we obtain a triangular matrix where all entries on an above the main diagonal are 1. Finally for M_{DISJ_n} we may see this considering the following recursive expansion of M_{DISJ_n}:

$$M_{\mathrm{DISJ}_1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \ , \ M_{\mathrm{DISJ}_{n+1}} = \left[\begin{array}{c:c} M_{\mathrm{DISJ}_n} & M_{\mathrm{DISJ}_n} \\ \hdashline M_{\mathrm{DISJ}_n} & 0 \end{array}\right] \ .$$

Now let M be such a triangular $m \times m$ matrix where all entries on the main diagonal are 1. If A is an equality–equivalent matrix to M then A would also be a triangular matrix where all entries on the main diagonal are nonzero. This implies the rank of A must be m. ■

Theorem 15. *Any* $\mathsf{ETHR} \circ \mathsf{SYM}$ *circuit computing either of the* $\overline{\mathrm{EQ}}$, GT, $\overline{\mathrm{GT}}$ *or* $\overline{\mathrm{DISJ}}$ *functions on $2n$ variables must be of size at least 2^n.* □

Corollary 16. $\mathsf{THR} \not\subseteq \mathsf{ETHR} \circ \mathsf{EMAJ}$, $\mathsf{EMAJ} \circ \mathsf{ETHR} \not\subseteq \mathsf{ETHR} \circ \mathsf{EMAJ}$, $\mathsf{MAJ} \circ \mathsf{MAJ} \not\subseteq \mathsf{ETHR} \circ \mathsf{EMAJ}$, $\mathsf{THR} \not\subseteq \mathsf{EMAJ} \circ \mathsf{EMAJ}$, $\mathsf{EMAJ} \circ \mathsf{ETHR} \not\subseteq \mathsf{EMAJ} \circ \mathsf{EMAJ}$.

Proof: The first separation holds because $\mathrm{GT} \in \mathsf{THR}$ and $\mathrm{GT} \notin \mathsf{ETHR} \circ \mathsf{EMAJ}$. The other separations holds because $\mathsf{THR} \subseteq \mathsf{EMAJ} \circ \mathsf{ETHR} \subseteq \mathsf{MAJ} \circ \mathsf{MAJ}$ and $\mathsf{EMAJ} \circ \mathsf{EMAJ} \subseteq \mathsf{ETHR} \circ \mathsf{EMAJ}$.

■

C. Closedness argument

Theorem 17. *The following separations holds:*

1) $\mathsf{THR} \circ \mathsf{MAJ} \neq \mathsf{ETHR} \circ \mathsf{ETHR}$.
2) $\mathsf{MAJ} \circ \mathsf{MAJ} \neq \mathsf{ETHR} \circ \mathsf{ETHR}$.
3) $\mathsf{THR} \circ \mathsf{MAJ} \neq \widehat{\mathsf{ELT}}_3$.

Proof: In this proof we will use the recent results that $\mathsf{AC}^0 \notin \mathsf{MAJ} \circ \mathsf{MAJ}$ ([8], [36]) and $\mathsf{AC}^0 \notin \mathsf{THR} \circ \mathsf{MAJ}$ ([34]). Our result follows immediately from the next two propositions. ■

Proposition 18. *The classes* $\mathsf{ETHR} \circ \mathsf{ETHR}$, $\mathsf{ETHR} \circ \mathsf{EMAJ}$, $\mathsf{EMAJ} \circ \mathsf{ETHR}$, *and* $\mathsf{EMAJ} \circ \mathsf{EMAJ}$ *are closed under conjunction. That is, if $\mathcal{C}$ denotes either of these classes of circuits we have* $\mathsf{AND} \circ \mathcal{C} = \mathcal{C}$.

Proof: We show it, say, for $\mathcal{C} = \mathsf{EMAJ} \circ \mathsf{ETHR}$. For the other classes the proof is similar. We have the following sequence of inclusions:

$$\mathsf{AND} \circ \mathsf{EMAJ} \circ \mathsf{ETHR} \subseteq$$
$$\mathsf{EMAJ} \circ \mathsf{AND}_2 \circ \mathsf{ETHR} \subseteq$$
$$\mathsf{EMAJ} \circ \mathsf{ETHR}.$$

Here the first inclusion follows by the third part of Proposition 6 and the second inclusion follows by the second part of the same proposition. ■

Proposition 19. *The classes* $\mathsf{MAJ}\circ\mathsf{MAJ}$ *and* $\mathsf{THR}\circ\mathsf{MAJ}$ *are not closed under conjunction, that is* $\mathsf{AND} \circ \mathsf{MAJ} \circ \mathsf{MAJ} \neq \mathsf{MAJ} \circ \mathsf{MAJ}$ *and* $\mathsf{AND} \circ \mathsf{THR} \circ \mathsf{MAJ} \neq \mathsf{THR} \circ \mathsf{MAJ}$.

Proof: We know that these classes do not contain AC^0. We also know that they are closed under negation (We even have $\mathsf{NOT}\circ\mathsf{MAJ} = \mathsf{MAJ}$ and $\mathsf{NOT}\circ\mathsf{THR} = \mathsf{THR}$). Assume now for contradiction that these classes are closed under conjunction. Since AC^0 can be generated by AND and NOT it would then follow the classes contained AC^0. ■

It is easy to see that by the same argument we can prove that $\mathsf{EMAJ} \circ \mathsf{ETHR} \neq \mathsf{MAJ} \circ \mathsf{MAJ}$. Since we know also that $\mathsf{EMAJ} \circ \mathsf{ETHR} \subseteq \mathsf{MAJ} \circ \mathsf{MAJ}$, it then follows that $\mathsf{MAJ} \circ \mathsf{MAJ} \nsubseteq \mathsf{EMAJ} \circ \mathsf{ETHR}$. But we actually can push this this argument further and get a simple concrete function separating the classes in this case.

Theorem 20. *We have* $\overline{\mathrm{DISJ}} \in \mathsf{MAJ} \circ \mathsf{MAJ}$*, but* $\overline{\mathrm{DISJ}} \notin \mathsf{EMAJ} \circ \mathsf{ETHR}$.

Proof: It is obvious from the definition of DISJ that $\overline{\mathrm{DISJ}}(x, y) = \bigvee_{i=1}^{n}(x_i \wedge y_i)$. Since $\mathrm{OR}, \mathrm{AND} \in \mathsf{MAJ}$ we have $\overline{\mathrm{DISJ}} \in \mathsf{MAJ} \circ \mathsf{MAJ}$. Razborov and Sherstov [34] proved that the function $\bigwedge_{j=1}^{n} \bigvee_{i=1}^{n}(x_{ji} \wedge y_{ji})$ is not in $\mathsf{THR}\circ\mathsf{MAJ}$ and hence is not in $\mathsf{EMAJ}\circ\mathsf{ETHR}$. By closedness of $\mathsf{EMAJ}\circ\mathsf{ETHR}$ under AND (Proposition 18) we then have that the function $\bigvee_{i=1}^{n}(x_i \wedge y_i)$ is also not in $\mathsf{EMAJ}\circ\mathsf{ETHR}$ and it is exactly the $\overline{\mathrm{DISJ}}$ function. ■

D. Separation from $\mathsf{MAJ} \circ \mathsf{MAJ}$

In this subsection we assume that Boolean variables range over $\{-1, 1\}$. That is, a Boolean function f is a function $f : \{-1, 1\}^n \to \{-1, 1\}$. It is easy to see that the same definitions of threshold and exact-threshold gates give us the same classes of Boolean circuits, so it does not matter in our consideration which values we associate to Boolean variables.

Definition 21. Let

$$P_n(x, y) = \sum_{i=0}^{n-1} \sum_{j=0}^{2n-1} 2^i y_j (x_{i,2j} + x_{i,2j+1})$$

and let $\mathrm{ep}_n(x, y) = -1$ if and only if $P_n(x, y) = -2$.

The next proposition was essentially proved by Goldmann, Håstad and Razborov [15].

$$\begin{array}{ccccc} & \mathsf{ETHR} \circ \mathsf{ETHR} & & & \\ \mathsf{ETHR} \circ \mathsf{EMAJ} & & & & \mathsf{EMAJ} \circ \mathsf{ETHR} \\ & \mathsf{EMAJ} \circ \mathsf{EMAJ} & & & \end{array}$$

Figure 4. Depth two exact threshold classes: All classes are distinct.

Proposition 22. $\mathrm{ep}_n(x, y) \notin \mathsf{MAJ} \circ \mathsf{MAJ}$

Proof: In [15] it is proved that the function $p_n(x, y) = \mathrm{sign}(2P_n(x, y) + 1)$ is not in $\mathsf{MAJ} \circ \mathsf{MAJ}$. But in the main part of the proof the authors restrict themselves to the inputs (x, y) such that $|P_n(x, y)| = 2$. On these inputs the function $p_n(x, y)$ is equivalent to $\mathrm{ep}_n(x, y)$. Thus it follows that also $\mathrm{ep}_n(x, y) \notin \mathsf{MAJ} \circ \mathsf{MAJ}$. ■

On the other hand it is easy to see that $\mathrm{ep}_n(x, y)$ is in $\mathsf{ETHR} \circ \mathsf{XOR}_2$ and thus it is in $\mathsf{ETHR} \circ \mathsf{EMAJ}$ (since this class is equal to $\mathsf{ETHR} \circ \mathsf{SYM}$). By this observation and by the inclusions proved above we have the following corollary:

Corollary 23. *We have* $\mathsf{ETHR} \circ \mathsf{EMAJ} \nsubseteq \mathsf{MAJ} \circ \mathsf{MAJ}$*,* $\mathsf{ETHR}\circ\mathsf{EMAJ} \nsubseteq \mathsf{EMAJ}\circ\mathsf{ETHR}$*,* $\mathsf{ETHR}\circ\mathsf{EMAJ} \nsubseteq \mathsf{EMAJ}\circ\mathsf{EMAJ}$*,* $\mathsf{ETHR} \circ \mathsf{ETHR} \nsubseteq \mathsf{MAJ} \circ \mathsf{MAJ}$*,* $\mathsf{ETHR} \circ \mathsf{ETHR} \nsubseteq \mathsf{EMAJ} \circ \mathsf{ETHR}$*,* $\mathsf{EMAJ} \circ \mathsf{EMAJ} \circ \mathsf{EMAJ} \nsubseteq \mathsf{MAJ} \circ \mathsf{MAJ}$.

We thus see that for depth two circuits we actually have a richer hierarchy (cf. Figure 4) for exact threshold circuits than for threshold circuits, since in the case of threshold circuits we have $\mathsf{MAJ} \circ \mathsf{THR} = \mathsf{MAJ} \circ \mathsf{MAJ}$ as given by Theorem 4.

V. The class AETM

Since we do not know an explicit lower bound for $\mathsf{ETHR}\circ\mathsf{ETHR}$, but we do know lower bounds for all its subclasses, it is natural to try to construct a new subclass containing all other subclasses discussed above. For this purpose we give the following definition:

Definition 24. $\mathsf{AETM} = (\mathsf{ETHR}\circ\mathsf{EMAJ})\wedge(\mathsf{EMAJ}\circ\mathsf{ETHR})$.

It might seem more natural to consider a conjunction of not just two classes of the sort $\mathsf{ETHR} \circ \mathsf{EMAJ}$ or $\mathsf{EMAJ} \circ \mathsf{ETHR}$, but of an arbitrary number of them. But from Proposition 18 it is easy to see that such a definition is equivalent to the above. The same proposition also shows that that the class AETM is closed under conjunction.

The next theorem positions the class AETM in our existing hierarchy.

Theorem 25. *The following inclusions holds:*

1) a) $\mathsf{ETHR} \circ \mathsf{EMAJ} \subseteq \mathsf{AETM}$
   b) $\mathsf{EMAJ} \circ \mathsf{ETHR} \subseteq \mathsf{AETM}$;
2) a) $\mathsf{AETM} \subseteq \mathsf{ETHR} \circ \mathsf{ETHR}$
   b) $\mathsf{AETM} \subseteq \mathsf{THR} \circ \mathsf{MAJ}$.

Proof: The first part of the theorem is obvious. For the first inclusion of the second part we have

$$\begin{aligned}&(\mathsf{ETHR}\circ\mathsf{EMAJ})\wedge(\mathsf{EMAJ}\circ\mathsf{ETHR})\subseteq\\&(\mathsf{ETHR}\circ\mathsf{ETHR})\wedge(\mathsf{ETHR}\circ\mathsf{ETHR})\subseteq\\&\mathsf{ETHR}\circ\mathsf{ETHR}\ .\end{aligned}$$

To prove the last inclusion of the theorem note that

$$\mathsf{ETHR}\wedge\mathsf{THR}\subseteq\mathsf{THR}\circ\mathsf{AND}_2\ .$$

Indeed, let the ETHR gate correspond to the equation $L_1(x)=0$ and the THR gate correspond to the inequality $L_2(x)\geq 0$. Let C be a constant such that for all x we have $|L_2(x)|<C$. Then the conjunction of $L_1(x)=0$ and $L_2(x)\geq 0$ is equivalent to $(-CL_1^2(x)+L_2(x)\geq 0)$.

Now we have

$$\begin{aligned}&(\mathsf{ETHR}\circ\mathsf{EMAJ})\wedge(\mathsf{EMAJ}\circ\mathsf{ETHR})\subseteq\\&(\mathsf{ETHR}\circ\mathsf{EMAJ})\wedge(\mathsf{MAJ}\circ\mathsf{THR})=\\&(\mathsf{ETHR}\circ\mathsf{EMAJ})\wedge(\mathsf{MAJ}\circ\mathsf{MAJ})=\\&(\mathsf{ETHR}\circ\mathsf{EMAJ})\wedge(\mathsf{MAJ}\circ\mathsf{EMAJ})\subseteq\\&\mathsf{THR}\circ\mathsf{AND}_2\circ\mathsf{EMAJ}=\\&\mathsf{THR}\circ\mathsf{EMAJ}=\mathsf{THR}\circ\mathsf{MAJ}.\end{aligned}$$

■

It is easy to separate AETM from $\mathsf{ETHR}\circ\mathsf{EMAJ}$ and $\mathsf{EMAJ}\circ\mathsf{ETHR}$ since we know functions that separates $\mathsf{ETHR}\circ\mathsf{EMAJ}$ and $\mathsf{EMAJ}\circ\mathsf{ETHR}$ from each other.

By the argument analogous to the Theorem 20 we can prove that $\overline{\mathrm{DISJ}}\notin\mathsf{AETM}$. Thus we have that $\mathsf{THR}\circ\mathsf{MAJ}\not\subseteq\mathsf{AETM}$ and $\mathsf{MAJ}\circ\mathsf{MAJ}\not\subseteq\mathsf{AETM}$. We do not know whether AETM is different from $\mathsf{ETHR}\circ\mathsf{ETHR}$, however.

VI. Conclusion.

The major open problem arising from our paper is to prove a strong lower bound for $\mathsf{ETHR}\circ\mathsf{ETHR}$ circuits. This question seems to be easier than the analogous question for $\mathsf{THR}\circ\mathsf{THR}$ — we conjecture that $\mathsf{ETHR}\circ\mathsf{ETHR}$ is a proper subclass of $\mathsf{THR}\circ\mathsf{THR}$, in the same way that the classes $\mathsf{EMAJ}\circ\mathsf{EMAJ}$, $\mathsf{EMAJ}\circ\mathsf{ETHR}$ and $\mathsf{ETHR}\circ\mathsf{EMAJ}$ are proper subclasses of the corresponding classes defined by threshold gates instead, $\mathsf{MAJ}\circ\mathsf{MAJ}$, $\mathsf{MAJ}\circ\mathsf{THR}$ and $\mathsf{THR}\circ\mathsf{MAJ}$.

We know lower bounds for subclasses of $\mathsf{ETHR}\circ\mathsf{ETHR}$. However we do not know whether the largest of those, AETM and $\mathsf{ETHR}\circ\mathsf{ETHR}$, are different. We believe that proving such a separation could give fruitful insight into $\mathsf{ETHR}\circ\mathsf{ETHR}$ circuits.

A similar question that could give fruitful insight into $\mathsf{THR}\circ\mathsf{THR}$ circuits would be to separate $\mathsf{THR}\circ\mathsf{THR}$ from $\mathsf{THR}\circ\mathsf{MAJ}$.

Acknowledgment

The work of the first author was supported by a postdoc fellowship from the Carlsberg Foundation. The work of the second author was partially supported by grant 09-01-12163-ofi_m from the Russian Foundation for Basic Research Fund. Part of this work was done during a visit to Aarhus University by the second author.

References

[1] K. Amano and A. Maruoka, "On the complexity of depth-2 circuits with threshold gates," in *Proceedings of the 30th International Symposium on Mathematical Foundations of Computer Science*, ser. Lecture Notes in Computer Science, vol. 3618. Springer, 2005, pp. 107–118.

[2] L. Babai, P. Frankl, and J. Simon, "Complexity classes in communication complexity theory," in *Proceedings of the 27th Annual IEEE Symposium on Foundations of Computer Science*. IEEE, 1986, pp. 337–347.

[3] L. Babai, K. A. Hansen, V. V. Podolskii, and X. Sun, "Weights of exact threshold functions," 2010, manuscript.

[4] R. Beigel and J. Tarui, "On ACC," *Computational Complexity*, vol. 4, no. 4, pp. 350–366, 1994.

[5] R. Beigel, J. Tarui, and S. Toda, "On probabilistic ACC circuits with an exact-threshold output gate," in *Proceedings of the Third International Symposium on Algorithms and Computation*, ser. Lecture Notes in Computer Science, vol. 650. Springer, 1992, pp. 420–429.

[6] J. Bourgain, "Estimation of certain exponential sums arising in complexity theory," *C. R. Acad. Sci. Paris, Ser. I*, vol. 341, pp. 627–631, 2005.

[7] J. Bruck, "Harmonic analysis of polynomial threshold functions," *SIAM Journal on Discrete Mathematics*, vol. 3, no. 2, pp. 168–177, 1990.

[8] H. Buhrman, N. K. Vereshchagin, and R. de Wolf, "On computation and communication with small bias," in *Proceedings of the 22nd Annual IEEE Conference on Computational Complexity*. IEEE Computer Society, 2007, pp. 24–32.

[9] A. K. Chandra, L. Stockmeyer, and U. Vishkin, "Constant depth reducibility," *SIAM Journal on Computing*, vol. 13, no. 2, pp. 423–439, 1984.

[10] A. Chattopadhyay and A. Wigderson, "Linear systems over composite moduli," in *Proceedings of the 50th Annual IEEE Symposium on Foundations of Computer Science*. IEEE, 2009, pp. 43–52.

[11] P. Erdös and J. Spencer, *Probabilistic Methods in Combinatorics*. Academic Press, 1974.

[12] J. Forster, "A linear lower bound on the unbounded error probabilistic communication complexity," *Journal of Computer and System Sciences*, vol. 65, no. 4, pp. 612–625, 2002.

[13] J. Forster, M. Krause, S. V. Lokam, R. Mubarakzjanov, N. Schmitt, and H.-U. Simon, "Relations between communication complexity, linear arrangements, and computational complexity," in *Proceedings of the 21st Conference on Foundations of Software Technology and Theoretical Computer Science*, ser. Lecture Notes in Computer Science, vol. 2245. Springer, 2001, pp. 171–182.

[14] M. Goldmann, "On the power of a threshold gate at the top," *Information Processing Letters*, vol. 63, no. 6, pp. 287–293, 1997.

[15] M. Goldmann, J. Håstad, and A. A. Razborov, "Majority gates vs. general weighted threshold gates," *Computational Complexity*, vol. 2, no. 4, pp. 277–300, 1992.

[16] M. Goldmann and M. Karpinski, "Simulating threshold circuits by majority circuits," *SIAM Journal on Computing*, vol. 27, no. 1, pp. 230–246, 1998.

[17] F. Green, "A complex-number fourier technique for lower bounds on the mod-m degree," *Computational Complexity*, vol. 9, no. 1, pp. 16–38, 2000.

[18] F. Green, A. Roy, and H. Straubing, "Bounds on an exponential sum arising in boolean circuit complexity," *C. R. Acad. Sci. Paris, Ser. I*, vol. 341, pp. 279–282, 2005.

[19] V. Grolmusz, "A lower bound for depth-3 circuits with MOD m gates," *Information Processing Letters*, vol. 67, no. 2, pp. 87–90, 1998.

[20] A. Hajnal, W. Maass, P. Pudlák, M. Szegedy, and G. Turán, "Threshold circuits of bounded depth," *Journal of Computer and System Sciences*, vol. 46, no. 2, pp. 129–154, 1993.

[21] K. A. Hansen, "Computing symmetric boolean functions by circuits with few exact threshold gates," in *Proceedings of the 13th Annual International Conference on Computing and Combinatorics*, ser. Lecture Notes in Computer Science, vol. 4598. Springer, 2007, pp. 448–458.

[22] ——, "Depth reduction for circuits with a single layer of modular counting gates," in *Proceedings of the 4th International Computer Science Symposium in Russia*, ser. Lecture Notes in Computer Science, vol. 5675. Springer, 2009, pp. 117–128.

[23] J. Håstad and M. Goldmann, "On the power of small-depth threshold circuits," *Computational Complexity*, vol. 1, pp. 113–129, 1991.

[24] J. Hertz, A. Krogh, and R. G. Palmer, *Introduction to the Theory of Neural Computation*. Addison-Wesley Publishing Company, 1991.

[25] T. Hofmeister, "A note on the simulation of exponential threshold weights," in *Proceedings of the 2nd Annual International Conference on Computing and Combinatorics*, ser. Lecture Notes in Computer Science. Springer, 1996, vol. 1090, pp. 136–141.

[26] M. Krause and P. Pudlák, "On the computational power of depth-2 circuits with threshold and modulo gates," *Theoretical Computer Science*, vol. 174, no. 1–2, pp. 137–156, 1997.

[27] M. Krause and S. Waack, "Variation ranks of communication matrices and lower bounds for depth-two circuits having nearly symmetric gates with unbounded fan-in," *Mathematical Systems Theory*, vol. 28, no. 6, pp. 553–564, 1995.

[28] M. L. Minsky and S. A. Papert, *Perceptrons: Expanded edition*. MIT Press, 1988.

[29] S. Muroga, *Threshold Logic and its Applications*. John Wiley & Sons, Inc., 1971.

[30] S. Muroga, I. Toda, and S. Takasu, "Theory of majority decision elements," *Journal of the Franklin Institute*, vol. 271, pp. 376–418, 1961.

[31] I. Parberry, *Circuit Complexity and Neural Networks*. Cambridge, MA: MIT Press, 1994.

[32] I. Parberry and G. Schnitger, "Parallel computation with threshold functions," *Journal of Computer and System Sciences*, vol. 36, no. 3, pp. 278–302, 1988.

[33] N. Pippenger, "The complexity of computations by networks," *IBM Journal of Research and Development*, vol. 31, no. 2, pp. 235–243, 1987.

[34] A. A. Razborov and A. A. Sherstov, "The sign-rank of AC^0," in *Proceedings of the 49th Annual IEEE Symposium on Foundations of Computer Science*. IEEE Computer Society, 2008, pp. 57–66.

[35] V. P. Roychowdhury, A. Orlitsky, and K.-Y. Siu, "Lower bounds on threshold and related circuits via communication complexity," *IEEE Transactions on Information Theory*, vol. 40, no. 2, pp. 467–474, 1994.

[36] A. A. Sherstov, "Separating AC^0 from depth-2 majority circuits," *SIAM J. Comput.*, vol. 38, no. 6, pp. 2113–2129, 2009.

[37] K.-Y. Siu and J. Bruck, "On the power of threshold circuits with small weights," *SIAM Journal on Discrete Mathematics*, vol. 4, no. 3, pp. 423–435, 1991.

[38] A. C.-C. Yao, "On ACC^0 and threshold circuits," in *Proceedings 31st Annual Symposium on Foundations of Computer Science*. IEEE Computer Society Press, 1990, pp. 619–627.