# Cubical sets as a classifying topos*

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**Abstract**

Coquand's cubical set model for homotopy type theory provides the basis for a computational interpretation of the univalence axiom and some higher inductive types, as implemented in the *cubical* proof assistant. We show that the underlying cube category is the opposite of the Lawvere theory of De Morgan algebras. The topos of cubical sets itself classifies the theory of 'free De Morgan algebras'. This provides us with a topos with an internal 'interval'. Using this interval we construct a model of type theory following van den Berg and Garner. We are currently investigating the precise relation with Coquand's. We do not exclude that the interval can also be used to construct other models.

## The topos of cubical sets

Simplicial sets from a standard framework for homotopy theory. The topos of simplicial sets is the classifying topos of the theory of strict linear orders with endpoints. Cubical sets turn out to be more amenable to a constructive treatment of homotopy type theory. Grandis and Mauri [GM03] describe the classifying theories for several cubical sets without diagonals. We consider the most recent cubical set model [Coq15]. This consists of symmetric cubical sets with connections $(\wedge, \vee)$, reversions $(\neg)$ and diagonals. Let $\mathbb{F}$ be the category of finite sets with all maps. Consider the monad $DM$ on $\mathbb{F}$ which assigns to each finite set $F$ the *finite* free DM-algebra (=De Morgan-algebra) on $F$. That this set is finite can be seen using the disjunctive normal form. The *cube category* in [Coq15] is the Kleisli category for the monad $DM$.

**Lawvere theory** Recall that for each algebraic (=finite product) theory $T$, the Lawvere theory $C_{fp}[T]$ is the opposite of the category of free finitely generated models. This is the classifying category for $T$: models of $T$ in any finite product category category $E$ correspond to product-preserving functors $C_{fp}[T] \to E$. The Kleisli category $KL_{DM}$ is precisely the *opposite* of the Lawvere theory for DM-algebras: maps $I \to DM(J)$ are equivalent to DM-maps $DM(I) \to DM(J)$ since each such DM-map is completely determined by its behavior on the atoms, as $DM(I)$ is free.

**Classifying topos** To obtain the classifying topos for an algebraic theory, we first need to complete with finite limits, i.e. to consider the category $C_{fl}$ as the *opposite* of finitely *presented* DM-algebras. Then $C_{fl}^{op} \to Set$, i.e. functors on finitely presented $T$-algebras, is the classifying topos. This topos contains a generic $T$-algebra $M$. $T$-algebras in any topos $\mathcal{F}$ correspond to *left exact left adjoint* functors from the classifying topos to $\mathcal{F}$.

Let $FG$ be the category of *free finitely generated* DM-algebras and let $FP$ the category of *finitely presented* ones. We have a fully faithful functor $f : FG \to FP$. This gives a geometric morphism $\phi$ between the functor toposes. Since $f$ is fully faithful, $\phi$ is an embedding.

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The subtopos $Set^{FG}$ of the classifying topos for DM-algebras is given by a quotient theory, the theory of the model $\phi^* M$. This model is given by pullback and thus is equivalent to the canonical DM-algebra $\mathbb{I}(m) := m$ for each $m \in FG$. So cubical sets are the classifying topos for 'free DM-algebras'. Each finitely generated DM-algebra has the disjunction property and is strict, $0 \neq 1$. These properties are geometric and hence also hold for $\mathbb{I}$. This disjunction property is important in the implementation [Coq15, 3.1].

This result can be generalized to related algebraic structures, e.g. Kleene algebras. A Kleene algebra is a DM-algebra with the property for all $x, y$, $x \wedge \neg x \leq y \vee \neg y$. With Coquand we checked that free finitely generated Kleene algebras also have the disjunction property.

## Model of type theory

Coquand's presentation of the cubical model does not build on a general categorical framework for constructing models of type theory. Docherty [Doc14] presents a model on cubical sets with connections using the general theory of path object categories [vdBG12]. The precise relation with the model in [BCH14] is left open. We present a slightly different construction using similar tools, but combined with internal reasoning. We use $\mathbb{I}$ as an 'interval'. To obtain a model of type theory on a category $C$ it suffices to provide an involutive 'Moore path' category object on $C$ with certain properties. Now, category objects on cubical sets are categories in that topos. The Moore path category $MX$ consists of lists of composable paths $\mathbb{I} \to X$ with the zero-length paths $e_x$ as left and right identity. To obtain a *nice* path object category, we quotient by the relation which identifies constant paths of any length. The reversion $\neg$ on $\mathbb{I}$ allows us to reverse paths of length 1. This reversion extends to paths of any length. We obtain an involutive category: Moore paths provide strictly associative composition, but non-strict inverses.

A path contraction is a map $MX \to MMX$ which maps a path $p$ to a path from $p$ to the constant path on $tp$ ($t$ for target). Like Docherty, we use connections to first define the map from $X^{\mathbb{I}}$ to $X^{\mathbb{I}\times\mathbb{I}}$ by $\lambda p.\lambda ij.p(i \vee j)$ and then extended this to a contraction. All these constructions are algebraic and hence work functorially. We obtain a nice path object category.

We have obtained a model of type theory [vdBG12, Doc14] starting from $\mathbb{I}$ in the cubical model. We plan to compare this more carefully with the one in [Coq15]. Like Voevodsky [Voe13], we define intensional identity types inside the extensional type theory of a topos.

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# References

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