Constructive algebra and geometric mathematics

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Some constructive algebra

Dynamical methods in algebra (Coste, Lombardi, Roy).
Inspired by topos theory and computer algebra (D5).

Hilbert program (Coquand/Lombardi):
Motivated by Barr's theorem, classifying topos
Extract constructive content from classical proofs.

Banachewski/Mulvey: Use of Barr's theorem in functional analysis.
Coquand/S: Direct constructive proofs using the methodology of constructive algebra.

Point-free Topology

The axiom of choice is used to construct ideal points (e.g. max. ideals).
Avoiding points one can avoid choice and non-constructive reasoning (Joyal, Mulvey, Coquand).
Slogan: using the axiom of choice is a choice!
Examples: Tychonoff, Krein-Millman, Alaoglu, Hahn-Banach, Gelfand, Zariski, ...

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Barr's Theorem: If a geometric sentence is deducible from a geometric theory in classical logic, with the axiom of choice, then it is also deducible from it constructively.
Classifying topos can often be constructed explicitly.

Adjunction

Frame: complete lattice where $\wedge$ distributes over $\bigvee$.
Morphisms preserve $\wedge, \bigvee$.
Locales are the opposite category of frames

There is an adjunction between spaces and locales.
With choice this restrict to an equivalence on compact Hausdorff spaces and compact regular locales.
A formula is positive when it uses only $\wedge, \bigvee$.
A geometric formula is an implication between to positive formulas.

Geometric type theory

We want to avoid infinitary logic($\bigvee$), not absolute.
Generalise to predicate logic.
Instead, define some geometric types (Vickers):

- free algebras: $\mathbb{N}$, list(A), $\mathcal{F}A$, ...
- coproducts, coequalizers, e.g. quotients: $\mathbb{Z}, \mathbb{Q}$, ...
- free models of Cartesian theories (=partial Horn logic), e.g. syntax of type theory, ...
- ...

Those constructions are preserved by geometric morphisms.
Importantly, not the power set!

Question

Geometric type theory has no precise syntax yet.

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Cut elimination (Negri) for first order geometric logic:
once formulated geometrically, proofs are usually natural and concise.
Even for theorems in higher order logic.
How do these works relate to geometric type theory?

Dedekind reals

Reals as a geometric theory with the natural topology

- $(\exists q : Q)L(q)$
- $(\forall q, q' : Q)(q < q' \wedge L(q') \rightarrow L(q))$
- $(\forall q : Q)(L(q) \rightarrow (\exists q' : Q)(q < q' \wedge L(q')))$
- $(\exists r : Q)R(r)$
- $(\forall r, r' : Q)(r' < r \wedge R(r') \rightarrow R(r))$
- $(\forall r : Q)(R(r) \rightarrow (\exists r' : Q)(r' < r \wedge R(r')))$
- $(\forall q : Q)(L(q) \wedge R(q) \rightarrow \perp)$
- $(\forall q, r : Q)(q < r \rightarrow L(q) \vee R(r))$

Commutative C*-algebras

For $X \in \mathbf{CptHd}$, consider $C(X, \mathbb{C})$.

It is a complex vector space:	$(f+g)(x) := f(x) + g(x)$, $(z \cdot f)(x) := z \cdot f(x)$.
It is a complex associative algebra:	$(f \cdot g)(x) := f(x) \cdot g(x)$.
It is a Banach algebra:	$\|f\| := \sup\{\|f(x)\| \ : \ x \in X\}$.
It has an involution:	$f^*(x) := \overline{f(x)}$.
It is a C*-algebra:	$\|f^* \cdot f\| = \|f\|^2$.
It is a commutative C*-algebra:	$f \cdot g = g \cdot f$.

In fact, X can be reconstructed from $C(X)$:
one can trade topological structure for algebraic structure.

Gelfand duality

There is a categorical equivalence (Gelfand duality):

$$\mathbf{CommC}^* \underset{C(-,\mathbb{C})}{\overset{\Sigma}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathbf{CptHd}^{\mathrm{op}}$$

The structure space $\Sigma(A)$ is called the Gelfand spectrum of A.

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To avoid choice want to define the spectrum geometrically.

Riesz space

The self-adjoint ('real') part of a C*-algebra is a Riesz space.

Definition

A *Riesz space* (vector lattice) is a vector space with 'compatible' lattice operations $\vee, \wedge$.
E.g. $f \vee g + f \wedge g = f + g$.

We assume that Riesz space R has a strong unit 1: $\forall f \exists n. f \leq n \cdot 1$.
Prime ('only') example:
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A representation of a Riesz space is a Riesz homomorphism to $\mathbb{R}$.
The representations of the Riesz space $C(X, \mathbb{R})$ are $\hat{x}(f) := f(x)$.

Theorem (Classical Stone-Yosida)

Let R be a Riesz space. Let $Max(R)$ be the space of representations. The space $Max(R)$ is compact Hausdorff and there is a Riesz embedding $\hat{\cdot} : R \to C(Max(R))$. The uniform norm of $\hat{a}$ equals the norm of a.

Formal space $Max(R)$

Geometric theory of representations

$D(a)$ '=' $\{\phi \in Max(R) : \hat{a}(\phi) > 0\}$. $a \in R$, $\hat{a}(\phi) = \phi(a)$

1. $D(a) \wedge D(-a) = 0$;
   $(D(a), D(-a) \vdash \bot)$
2. $D(a) = 0$ if $a \leq 0$;
3. $D(a+b) \leq D(a) \vee D(b)$;
4. $D(1) = 1$;
5. $D(a \vee b) = D(a) \vee D(b)$
6. $D(a) = \bigvee_{r>0} D(a-r)$.

$Max(R)$ is compact completely regular (cpt Hausdorff)
Pointfree description of the space of representations $Max(R)$
'Every Riesz space is a Riesz space of functions'
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Retract

Every compact regular space X is retract of a coherent space Y
$f : Y \twoheadrightarrow X$, $g : X \rightarrowtail Y$, st $f \circ g = \mathrm{id}$ in Loc
$f : X \twoheadrightarrow Y$, $g : Y \rightarrowtail X$, st $g \circ f = \mathrm{id}$ in Frm
Strategy: first define a finitary cover, then add the infintary part and prove that it is a conservative extension. (Coquand, Mulvey)

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Above: The interpretation $D(a) := \bigvee_{r>0} D(a - r)$ defines a embedding $g : Y \rightarrowtail X$ in Frm validating axiom 6
Finitary proof of Stone-Yosida and Gelfand duality.

Coquand's normal distributive lattices

Compact regular Loc are equivalent to compact Hausdorff Sp
The finite cover can be presented by normal DL.

$a' \preccurlyeq a$ (a' well inside a) if there is y s.t. $a \vee y = \top$ and $a \wedge y = \bot$.
L is *normal* if $a \vee b = \top$, then there is $a' \preccurlyeq a$ with $a' \vee b = \top$.
A prime filter $F \subseteq L$ is *regular* if whenever $a \in F$, then there exists $a' \preccurlyeq a$ s.t. $a \in F$.

The points of a compact regular locale are its regular filters.

Coquand's normal distributive lattices

In fact, the frame itself is geometric!
The theory of RSpec regular filters is geometric.
Geometric theory RIdl of the $\leqslant$-rounded ideals.
The points of RIdl form the frame of opens.

Theorem (S/Vickers/Wolters)

Let L be a normal distributive lattice. Then $\mathrm{RIdl}L = \mathbb{S}^{\mathrm{RSpec}L}$. The evaluation map $\times \to \mathbb{S}$ takes (I, x) to $\top$ whenever the ideal I meets the filter x.

$\mathbb{S}$ is Sierpinsky space.

Facilitates computations in quantum topos theory.

Bohr toposes

Relate algebraic quantum mechanics to topos theory
to construct new kind of quantum spaces.
— A spectral invariant for non-commutative algebras —

Classical physics

Standard presentation of classical physics:
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An observable a and an interval $\Delta \subseteq \mathbb{R}$ together define a *proposition* '$a \in \Delta$' by the set $a^{-1}\Delta$.

Classical (sets) or geometric (spaces) logic

Quantum

How to generalize to the quantum setting?

1. Identifying a quantum phase space Σ.
2. Defining subsets of Σ acting as propositions of quantum mechanics.
3. Describing states in terms of Σ.

Old-style quantum logic

von Neumann proposed:

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Von Neumann later abandoned this.
No implication, no deductive system.

Bohrification

In classical physics we have a spatial logic.
Want the same for quantum physics. So we consider two generalizations of topological spaces:

- C*-algebras (Connes' non-commutative geometry)
- toposes and locales (Grothendieck)

We connect the two generalizations by:

1. *Algebraic quantum theory*
2. *Constructive Gelfand duality*
3. *Bohr's doctrine of classical concepts*

[Heunen, Landsman, S]

C*-algebras

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Prime example:
$B(H) = \{f : H \to H \mid f \text{ bounded linear}\}$, for H Hilbert space.

is a complex vector space:	$(f+g)(x) := f(x) + g(x)$, $(z \cdot f)(x) := z \cdot f(x)$,
is an associative algebra:	$f \cdot g := f \circ g$,
is a Banach algebra:	$\|f\| := \sup\{\|f(x)\| \ : \ \|x\| = 1\}$,
has an involution:	$\langle fx, y \rangle = \langle x, f^* y \rangle$
satisfies:	$\|f^* \cdot f\| = \|f\|^2$,
but not necessarily:	$f \cdot g = g \cdot f$.

Classical concepts

Bohr's "doctrine of classical concepts" states that we can only look at the quantum world through classical glasses, measurement merely providing a "classical snapshot of reality". The combination of all such snapshots should then provide a complete picture.

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Let A be a C*-algebra (quantum system)
The set of as 'classical contexts', 'windows on the world':

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A is not entirely determined by $\mathcal{C}(A)$: $\mathcal{C}(A) = \mathcal{C}(A^{\mathrm{op}})$

Doering and Harding, Hamhalter
the Jordan structure can be retrieved.

Internal C*-algebra

Internal reasoning: topological group = group object in Top.
Internal C*-algebras in $\mathbf{Set}^{\mathbf{C}}$ are functors of the form $\mathbf{C} \to \mathbf{CStar}$.
'Bundle of C*-algebras'.

We define the Bohrification of A as the internal C*-algebra

$$\underline{A} : \mathcal{C}(A) \to \mathbf{Set},$$
$$V \mapsto V.$$

in the topos $\mathcal{T}(A) = \mathbf{Set}^{\mathcal{C}(A)}$, where
$\mathcal{C}(A) := \{V \subseteq A \mid V \text{ commutative C*-algebra}\}$.

The internal C*-algebra $\underline{A}$ is commutative!
This reflects our Bohrian perspective.

Kochen-Specker

Theorem (Kochen-Specker): no hidden variables in QM.
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Isham-Döring: a certain *global* section does not exist.
We can still have neo-realistic interpretation by considering also non-global sections.
In quantum gravity there can be no external observer.
In fact, algebraic quantum field theory provides a topos with an *internal* C*-algebra.

Phase object in a topos

Apply constructive Gelfand duality (Banachewski, Mulvey) to the Bohrification to obtain the (internal) spectrum Σ.
This is our phase object. (motivated by Döring-Isham).

Kochen-Specker $=$ Σ has no (global) point.
However, Σ is a well-defined interesting compact regular locale.
Pointless topological space of hidden variables.

Externalizing

$PSh(P) \equiv Sh(\mathrm{Idl}P)$, Scott topology.
$Loc_{Sh(X)} \equiv Loc_{/X}$
There is an external locale $\Sigma \to \mathrm{Idl}(\mathcal{C}(A))$ equivalent to $\underline{\Sigma}$ in $\mathcal{T}(A)$
When applied to the lattice of projections of a Hilbert space we turn old style quantum logic into a Heyting algebra.

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Our definition of the spectrum is *geometric*.
Hence, Σ can be computed fiberwise: points (C, σ)

Points

Is Σ spatial (have enough points)?

1. Yes, frame of a topological space
2. It is constructively locally compact!

2a. Σ is compact regular in $\mathrm{Sh}(\mathrm{Idl}(\mathcal{C}(A)))$

2b. $\mathrm{Idl}(\mathcal{C}(A))$ is locally compact (Scott domain)

2c. Locally compact maps compose

2d. Locally compact locales are classically spatial

Locally compact

$Loc_{Sh(X)} \equiv Loc_{/X}$
Hyland TFAE:

- Y locally compact
- The exponential $\mathbb{S}^Y$ exists; $\mathbb{S}$=Sierpiński locale
- Y is exponentiable

Theorem: If $Y \to X$ locally compact in $Sh(X)$, X locally compact. Then Y is locally compact.

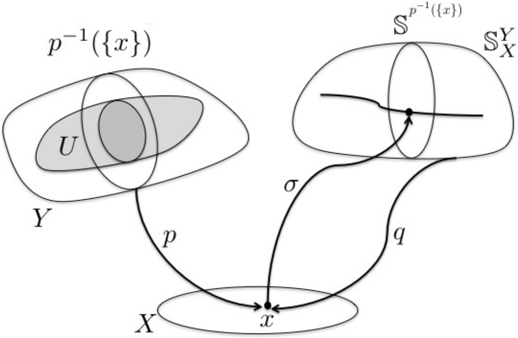
Geometric logic

Constructive transformation of points from X to Y gives a locale map from X to Y, even if X, Y are not spatial.

Locally compact

Theorem: $Y \to X$ locally compact in $Sh(X)$, X locally compact. Then Y is locally compact.

Proof: Need to construct $\mathbb{S}^Y$ (opens of Y).



Locales by geometric theories

Continuous map: constructive transformations of points

Continuous map as a bundle

Locally perfect

Perfect maps correspond to internal compact locales
Locally perfect maps correspond to internal locally compact locales
New theorem in topology:
Locally perfect maps compose (needs some separation).
Corollary:
the external spectrum is locally compact and hence spatial

Conclusions

Application of constructive algebra to QM via topos theory.
Bohr's doctrine suggests a topos making a C*-algebra commutative

- Spatial quantum logic via topos logic
- Phase space via internal Gelfand duality
- Spectral invariant for non-commutative algebras.
- States (non-commutative integrals) become internal integrals.

Geometric mathematics makes computations manageable.